Interference between independent photons

H. Paul

Zentralinstitut für Optik und Spektroskopie, Akademie der Wissenschaften der DDR, DDR-1199 Berlin, German Democratic Republic

A comprehensive review is given of investigations, both theoretical and experimental, that demonstrate interference to take place even when the fields involved are produced by independent sources. The physical situation is rather different for coherent and incoherent fields. In the first case, where the sources are lasers, a conventional interference pattern can be observed which is adequately described by the classical theory. In the present paper, special attention is paid to the question of whether this interference persists when the two laser beams become strongly attenuated. In the second case, the sources are individual atoms excited by a pumping mechanism, that emit spontaneously and, hence, independently from each other. In those circumstances, no interference pattern can show up. However, it becomes evident from both the classical and the quantum-mechanical theory that interference effects can still be established by observing intensity correlations rather than the intensity itself. This point is discussed in greater detail. The pioneering experiments of Forrester, Gudmundson and Johnson, and Brown and Twiss are reviewed in this context. Especially interesting from the theoretical point of view is the case of two emitting atoms, since then the classical and the quantum-mechanical description differ significantly, the quantum theory predicting the intensity correlations to be distinctly stronger than those following from classical considerations. This specific quantum-mechanical feature is shown to be intimately connected with the corpuscular aspect of light.

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I. INTRODUCTION

As early as 1930 Dirac, in his famous textbook on quantum mechanics (Dirac, 1930), made a basic statement on optical interference: "Each photon interferes only with itself. Interference between two different photons never occurs." While the first part of this assertion undoubtedly provides the correct quantum-mechanical interpretation of all conventional interference experiments, its second part cannot be upheld as a general rule forbidding independent photons, i.e., photons being emitted by independent sources, to interfere.

Of course, Dirac is right with respect to the experimental techniques that were at the optical researcher's disposal at that time. On the one hand, only light sources of a thermal nature (consisting of independent, "elementary" radiators) were available, and, on the other hand, interference could be detected only in the form of stationary patterns (by means of a photographic plate or the eye) indicative of inhomogeneous distributions of the light intensity in space. In these circumstances, only interference of the photon with itself can be observed; in other words, only light beams that originate from a common primary beam (produced with the help of a beam-splitting device like a half-silvered mirror or simply a screen with some little holes in it) can be made to interfere.

Since then, however, impressive progress in the development of both light sources and detectors has taken place. First, the laser was invented, a marvelous new light source that (for the first time in the history of optics) made possible the production of coherent radiation. Second, the advent of fast photodetectors made fluctuation phenomena occurring in optical fields accessible to observation. In fact, intensity correlations (both spatial and temporal) can be measured with these devices, as was demonstrated first by Brown and Twiss (1956a). Both inventions opened new areas of experimental investigation and, in particular, made possible novel types of interference experiments.

The availability of lasers naturally led to the question of whether the light beams from two lasers that are independently operated might interfere, the answer—as one might expect from classical electrodynamics, an affirmative one—being first given by Magyar and Mandel (1963). With the help of photocells or photomultipliers, on the other hand, correlation measurements can be performed that show interference to have taken place, even in situations where no interference pattern is observable.

Interference is thus understood in a more general sense than in classical optics, where it is synonymous with the

occurrence of a (stationary) intensity pattern or a beat signal produced by two coherent light waves differing a little in their frequencies. However, we should be aware of the fact that the reason for this conventional point of view is merely our inability to observe directly the electrical field strength residing in optical fields. Basically, interference means superposition of electromagnetic fields, and any experiment indicating that such a superposition has taken place deserves to be called an interference experiment. In this sense, the observation of photoelectric mixing of incoherent light by Forrester, Gudmundson, and Johnson (1955) and the detection of spatial intensity correlations in thermal light by Brown and Twiss (1956a) are the first experiments that provided evidence of interference between photons spontaneously emitted by different atoms. Hence the second part of Dirac's famous statement quoted above has actually been disproved.

It is the purpose of the present paper to give a comprehensive review of both the theoretical and the experimental work on interference between independent photons that has been carried out in the last thirty years. Emphasis will be placed on the physical concepts rather than formal considerations. Since interference is basically a classical phenomenon one encounters whenever one has to deal with waves (the interference between water waves, for instance, is something we experience in our childhood), it appears natural to start by reviewing the main features of the classical description of interference. This will be done in Sec. II. From the viewpoint of classical theory, the ideal conditions for interference to show up are provided by two coherent waves. Since these can nowadays be produced with the help of single-mode lasers, we are led to consider interference between independent lasers beams in Sec. III. In this context, the important question arises of whether this interference phenomenon will persist when the two laser beams are strongly attenuated, so that only a few photons are contained in the coherence volume. This problem will be treated in Sec. III.C, while Sec. IV deals with interference effects in incoherent light. The basic experiments are reviewed and a physical explanation is given. Further, it will be shown theoretically that interference between two photons emitted independently by two excited atoms becomes manifest in a modulation of the intensity correlation function. This case deserves special attention, since here the quantum-mechanical predictions differ distinctly from the classical ones.

II. CLASSICAL DESCRIPTION OF INTERFERENCE

A. Interference between coherent beams

As already mentioned in Sec. I, interference is to be understood quite generally as a superposition of waves, i.e., an algebraic addition of the corresponding "elongations" which, in optics, are to be specified as the electric and (with minor experimental relevance) the magnetic field strengths. It is obvious that in this process the phases of the waves play an important role. The situation is simplest, of course, when the phases do not noticeably vary in time. (Practically, we require them to be constant only over time intervals of the length of the observation time.) Then an interference pattern will show up in the intensity of the superposition field.

Let us consider, for simplicity, two monochromatic traveling plane waves linearly polarized in the same direction; we may write the positive-frequency part of the electric field strength in the two beams labeled j as

$$E_j^{(+)}(\mathbf{r},t)=E_j e^{i(\mathbf{k}_j\mathbf{r}-\omega_j t-\Phi_j)} \quad (j=\mathrm{I},\mathrm{II}) . \tag{2.1}$$

Here E_j (assumed real) is the amplitude, $\mathbf{k}_j$ the wave vector, ω_j the circular frequency, and Φ_j the phase. In a strictly monochromatic wave, both the amplitude and the phase are constants. The intensity, defined conveniently as one-half of the square of the electric field strength averaged over some oscillation periods, in the superposition field follows from Eq. (2.1):

$$\begin{aligned} I(\mathbf{r},t)=&E_{\mathrm{I}}^2+E_{\mathrm{II}}^2 \\ &+2E_{\mathrm{I}}E_{\mathrm{II}}\cos[(\mathbf{k}_{\mathrm{II}}-\mathbf{k}_{\mathrm{I}})\mathbf{r}-(\omega_{\mathrm{II}}-\omega_{\mathrm{I}})t \\ &-(\Phi_{\mathrm{II}}-\Phi_{\mathrm{I}})] . \end{aligned} \tag{2.2}$$

This well-known formula describes a standing (spatial) interference pattern for $\omega_{\mathrm{I}}=\omega_{\mathrm{II}}$. It should be emphasized that its position critically depends on the phase difference $\Delta\Phi=\Phi_{\mathrm{II}}-\Phi_{\mathrm{I}}$. On the other hand, for different frequencies $\omega_{\mathrm{II}}\neq\omega_{\mathrm{I}}$ a sinusoidal temporal modulation of the intensity at any given point in space is predicted.

These two phenomena, as simple as they appear from the theoretical viewpoint, nevertheless could not be observed until the advent of the laser, which, for the first time, made it possible to produce coherent light, i.e., light whose phase (and amplitude) remains constant in time intervals long enough for an actual observation to be made. The corresponding experiments performed in the early days of the laser era will be described in Sec. III.A.

B. Conventional interference experiments

Unlike laser radiation, light from conventional sources exhibits rapid fluctuations in both the amplitude and the phase of the electric field strength. In particular, the phase undergoes many random changes during the observation time. Hence two *independent* beams emitted from two sources (or two different parts of the same source) cannot produce an observable interference pattern, since according to Eq. (2.2) the instantaneous pattern—provided the intensity is so high that interference fringes will be formed at all[1] over the short time intervals in

[1]In the detection process the particle aspect of light becomes dominant. Hence an interference pattern can be formed only by a large number of photons that produce, for instance, blackened spots in a photographic plate.

which the phases remain constant—will be displaced, by a random fraction of the fringe spacing, whenever the phases $\Phi_{\rm I}$ and $\Phi_{\rm II}$ "jump."

There is, however, a way out of this problem. Since the interference term on the right-hand side of Eq. (2.2) depends only on the phase difference $\Delta\Phi=\Phi_{\rm II}-\Phi_{\rm I}$, only the phase difference is actually required to remain constant. This allows the phases themselves to fluctuate. However, the fluctuations of $\Phi_{\rm I}$ and $\Phi_{\rm II}$ must not be independent; instead, a definite correlation should be established between them such that one copies, more or less precisely, the other. Experimentally, this is achieved (e.g., by means of a beam splitter) by making the interfering beams replicas of only one primary beam. In fact, all conventional interference experiments are of this kind. However, care has to be taken that the path difference between two such beams, in the receiving plane, does not exceed the coherence length, since this length marks, roughly speaking, the distance (in the direction of beam propagation) over which the phase remains virtually constant, on the average. Hence the individual field strengths (residing at the same position) in the two beams will no longer be correlated when one of them is "shifted" with respect to the other by more than the coherence length.

From the quantum-mechanical point of view, the basic mechanism underlying conventional interference experiments is, in Dirac's words, "the interference of the photon with itself." In fact, it has been demonstrated experimentally by several authors [see, for instance, Taylor (1909); Jánossy and Náray (1957,1958)] that the interference phenomena still persist, without loss of visibility, down to intensities so extremely low that the time interval between the arrival of any photon and the next is much longer, on the average, than the transit time through the apparatus (e.g., a Fabry-Perot interferometer).

Hence the interference of the photon with itself is well established, both experimentally and theoretically. Since my attention is focused on interference between independent photons, I have mentioned only the main aspects of what is usually understood by interference in optics.

C. Intensity correlations

The development of photodetectors made possible the observation of intensity correlations in light fields (preferentially incoherent ones). Postponing a description of the experimental procedure to Secs. III.D, IV.A, and IV.B, I shall first discuss this subject from the viewpoint of classical theory.

Generally, the intensity correlation function is defined as the mean value of the product of two intensities $I(\mathbf{r}_1,t_1)$ and $I(\mathbf{r}_2,t_2)$

$$G^{(2)}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\langle\langle I(\mathbf{r}_1,t_1)I(\mathbf{r}_2,t_2)\rangle\rangle\ . \tag{2.3}$$

The averaging, indicated by the double angular bracket, is over an ensemble. (It is well known that for stationary fields the ensemble average is equivalent to the temporal average, as a consequence of the ergodic behavior of the field.)

What I want to show now is that specific intensity correlations exist even in situations where the mean intensity is constant in space, and hence no interference pattern is observable with the help of conventional techniques. Since they are brought about by superposition of different waves, these correlations provide indirect evidence that interference has taken place.

As before, let us consider the simple case of two plane waves with equal frequencies and polarization properties. For simplicity, we assume the amplitudes $E_{\rm I}$ and $E_{\rm II}$ to be fixed. The phases, however, fluctuate randomly and independently of one another. (In the formal description we thus have to deal with an ensemble of two-field systems with randomly distributed phases $\Phi_{\rm I}$ and $\Phi_{\rm II}$.)

Consequently, the ensemble average over the intensity is constant in space, as follows directly from Eq. (2.2). However, the intensity correlation function (2.3), taken for $t_1=t_2$, exhibits a distinct spatial modulation. In fact, one easily finds from Eqs. (2.2) and (2.3), putting $E_{\rm I}^2=I_{\rm I}$ and $E_{\rm II}^2=I_{\rm II}$,

$$\begin{aligned}G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)&=\langle\langle I(\mathbf{r}_1,t)I(\mathbf{r}_2,t)\rangle\rangle\\&=(I_{\rm I}+I_{\rm II})^2+2I_{\rm I}I_{\rm II}\cos[(\mathbf{k}_{\rm II}-\mathbf{k}_{\rm I})(\mathbf{r}_2-\mathbf{r}_1)]\ .\end{aligned} \tag{2.4}$$

In the special case of equal intensities in the two beams, $I_{\rm I}=I_{\rm II}$, Eq. (2.4) takes the simple form

$$\begin{aligned}&G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)\\&\quad=\langle\langle I\rangle\rangle^2\{1+\tfrac{1}{2}\cos[(\mathbf{k}_{\rm II}-\mathbf{k}_{\rm I})(\mathbf{r}_2-\mathbf{r}_1)]\}\ ,\end{aligned} \tag{2.5}$$

where $\langle\langle I\rangle\rangle=I_{\rm I}+I_{\rm II}=2I_{\rm I}$ is the mean intensity.

It is not difficult to take into account amplitude or intensity fluctuations, in addition to the phase fluctuations considered thus far, in the two beams. Assuming the amplitudes $E_{\rm I}$ and $E_{\rm II}$ to fluctuate independently of one another and of the phases, we can easily generalize Eq. (2.4) in the form

$$\begin{aligned}&G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)\\&\quad=\langle\langle I_{\rm I}^2\rangle\rangle+\langle\langle I_{\rm II}^2\rangle\rangle\\&\qquad+2\langle\langle I_{\rm I}\rangle\rangle\langle\langle I_{\rm II}\rangle\rangle\{1+\cos[(\mathbf{k}_{\rm II}-\mathbf{k}_{\rm I})(\mathbf{r}_2-\mathbf{r}_1)]\}\ .\end{aligned} \tag{2.6}$$

When the intensity fluctuations are of the type known from thermal light, $\langle\langle I_j^2\rangle\rangle$ is just twice the value of $\langle\langle I_j\rangle\rangle^2$ $(j=\mathrm{I,II})$. Then from Eq. (2.6) we obtain, for $\langle\langle I_1\rangle\rangle=\langle\langle I_2\rangle\rangle$,

$$G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)=\tfrac{3}{2}\langle\langle I\rangle\rangle^2\{1+\tfrac{1}{3}\cos[(\mathbf{k}_{\rm II}-\mathbf{k}_{\rm I})(\mathbf{r}_2-\mathbf{r}_1)]\}\ . \tag{2.7}$$

Obviously, the intensity correlation depends sensitively on the difference $\mathbf{r}_2-\mathbf{r}_1$. One learns from comparison of Eqs. (2.5) and (2.7) that this effect is most pronounced in the absence of intensity fluctuations in the individual beams. It is, nevertheless, noticeable also in the case of strong intensity fluctuations, as they are exhibited by thermal light.

Let us denote, for convenience, the direction of the vector $\mathbf{k}_2-\mathbf{k}_1$ as the z direction, which allows us to write $\Delta k(z_2-z_1)$ (where Δk is the difference of the z components of $\mathbf{k}_{II}$ and $\mathbf{k}_I$, respectively) instead of $(\mathbf{k}_{II}-\mathbf{k}_I)(\mathbf{r}_2-\mathbf{r}_1)$. All the correlation functions (2.4)—(2.7) take their maxima when

$$\Delta k(z_2-z_1)=0 \mod 2\pi , \tag{2.8}$$

whereas minima occur for

$$\Delta k(z_2-z_1)=\pi \mod 2\pi . \tag{2.9}$$

For a physical interpretation of these results, we need only look at Eq. (2.2) which describes the interference pattern showing up in the case of any fixed phase difference $\Phi_{II}-\Phi_I$. This formula indicates that the intensity pattern is periodic in z, the period being given by

$$\Lambda=\frac{2\pi}{\Delta k} . \tag{2.10}$$

In particular, Λ equals the distance, in the z direction, between two neighboring maxima of the (total) intensity, i.e., the fringe spacing.

Hence condition (2.8) means that the z coordinates of the two points $\mathbf{r}_1$ and $\mathbf{r}_2$ differ by just a multiple of the fringe spacing. Similarly, condition (2.9) requires that the difference between z_2 and z_1 coincide with an odd number of half-fringe spacings.

Intuitively, the variation of the intensity correlation functions (2.4)—(2.7) can be understood in the following manner. In case (2.8) remarkably large contributions to $G^{(2)}$ will come from those special realizations of the field (corresponding to an appropriate value of the phase difference) where one of the points $\mathbf{r}_1,\mathbf{r}_2$ lies in the neighborhood of an intensity maximum, since then, according to what has been said before, the other point will do the same. This will make the ensemble average of $I(\mathbf{r}_1,t)I(\mathbf{r}_2,t)$ larger than in case (2.9), where the situation is quite different: When the intensity happens to be maximum at $\mathbf{r}_1$, it will be just minimum at $\mathbf{r}_2$, so that the product of both intensities vanishes.

That $G^{(2)}$ takes, in fact, an absolute maximum, when condition (2.9) is fulfilled, can be seen from a quite general argument. Since the "instantaneous" interference pattern (2.2) is periodic in space, we are actually dealing with the correlation function $G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_1,t)$, even when z_2-z_1 is a nonzero multiple of the fringe spacing; it is well known from statistical communication theory that the (auto)correlation function has an absolute maximum when the two arguments coincide. This is readily proved [see, for example, Middleton (1960)] by starting from the relation

$$\langle\langle[I(\mathbf{r}_1,t)-I(\mathbf{r}_2,t)]^2\rangle\rangle\geq 0 , \tag{2.11}$$

which is certainly true. Evaluating the left-hand side of this inequality, one immediately finds, under the assumption $\langle\langle I^2(\mathbf{r}_1,t)\rangle\rangle=\langle\langle I^2(\mathbf{r}_2,t)\rangle\rangle$,

$$\langle\langle I^2(\mathbf{r}_1,t)\rangle\rangle\geq\langle\langle I(\mathbf{r}_1,t)I(\mathbf{r}_2,t)\rangle\rangle . \tag{2.12}$$

The variation of $G^{(2)}$ with $\mathbf{r}_2-\mathbf{r}_1$ is displayed even more distinctly when we consider the deviation from the mean intensity,

$$\Delta I(\mathbf{r},t)=I(\mathbf{r},t)-\langle\langle I\rangle\rangle , \tag{2.13}$$

rather than the intensity [note that the mean intensity $\langle\langle I\rangle\rangle$ is constant in space, as follows from Eq. (2.2)].

From Eq. (2.13) we find the correlation function to be

$$\langle\langle\Delta I(\mathbf{r}_1,t)\Delta I(\mathbf{r}_2,t)\rangle\rangle=\langle\langle I(\mathbf{r}_1,t)I(\mathbf{r}_2,t)\rangle\rangle-\langle\langle I\rangle\rangle^2 . \tag{2.14}$$

Utilizing Eq. (2.6) and noticing that $\langle\langle I\rangle\rangle=\langle\langle I_I\rangle\rangle+\langle\langle I_{II}\rangle\rangle$, we can rewrite Eq. (2.14) as

$$\langle\langle\Delta I(\mathbf{r}_1,t)\Delta I(\mathbf{r}_2,t)\rangle\rangle=\Delta I_I^2+\Delta I_{II}^2+2\langle\langle I_I\rangle\rangle\langle\langle I_{II}\rangle\rangle\cos[\Delta k(z_2-z_1)] , \tag{2.15}$$

where ΔI_j^2 $(j=\mathrm{I,II})$ is the mean-square deviation of the intensities in the individual beams, $\Delta I_j^2\equiv\langle\langle(I_j-\langle\langle I_j\rangle\rangle)^2\rangle\rangle=\langle\langle I_j^2\rangle\rangle-\langle\langle I_j\rangle\rangle^2$. When the intensities I_I and I_{II} are constant, Eq. (2.15) reduces to the simple form

$$\begin{aligned}\langle\langle\Delta I(\mathbf{r}_1,t)\Delta I(\mathbf{r}_2,t)\rangle\rangle&=2I_II_{II}\cos[\Delta k(z_2-z_2)]\\&=\tfrac{1}{2}\langle\langle I\rangle\rangle^2\cos[\Delta k(z_2-z_1)] \text{ for } I_I=I_{II} .\end{aligned} \tag{2.16}$$

This correlation function takes on positive as well as negative values. In the latter case one speaks of anticorrelations. Plainly these are strongest when the distance between z_1 and z_2 equals an odd half-number of fringe spacings.

Thus it has been shown that intensity correlations exhibit interference effects even in situations where no conventional interference pattern is observable. Certainly this feature will not be restricted to the plane-wave field configuration considered thus far. Of special physical interest will be the case of two or more spontaneously emitting atoms. We first discuss this problem in a classical model, postponing the quantum-mechanical treatment to Secs. IV.C and IV.D.

To this end, we calculate the intensity correlation function for the superposition field produced by two Hertzian oscillators, radiating with random phases. We separate the dipole moment oscillating at ω into a positive- and a negative-frequency part, $\mathbf{d}=\mathbf{d}^{(+)}+\mathbf{d}^{(-)}$, and set, for convenience,

$$\mathbf{d}^{(+)}(t)=\mathbf{d}a(t) , \tag{2.17}$$

where the vector $\mathbf{d}$ (assumed real) indicates both the direction and, through its length, the maximum amplitude of the dipole oscillation. As a result of radiation damping, the dipole, once put into oscillation at $t=0$ and afterwards left to itself, will be damped exponentially,

$$a(t)=e^{-i\omega t}\tilde{a}(t),\quad \tilde{a}(t)=e^{-(\Gamma/2)t} \text{ for } t\geq 0 . \tag{2.18}$$

Here we have taken the initial value $a(0)=\tilde{a}(0)$ as unity.

Later on, a (constant) phase factor $\exp(-i\Phi)$ will be attached explicitly.

According to the well-known theory of the Hertzian oscillator, the electric field strength in the far field can be written as [see Kimble and Mandel (1976)]

$$\mathbf{E}^{(+)}(\mathbf{r},t)=\mathbf{f}(\mathbf{r}-\mathbf{r}_0)a\left[t-\frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right], \tag{2.19}$$

where $\mathbf{r}_0$ is the position of the dipole and

$$\mathbf{f}(\mathbf{R})=\frac{\omega^2}{4\pi\varepsilon_0 c^2}\left[\frac{\mathbf{d}}{R}-\frac{(\mathbf{dR})\mathbf{R}}{R^3}\right](|\mathbf{R}|=R) \tag{2.20}$$

describes the spatial distribution of the dipole radiation.

The field originating from two dipoles located at $\mathbf{r}_\mathrm{I}$ and $\mathbf{r}_\mathrm{II}$, respectively, is thus given by

$$\mathbf{E}^{(+)}(\mathbf{r},t)=\mathbf{f}(\mathbf{r}-\mathbf{r}_\mathrm{I})a_\mathrm{I}\left[t-\frac{|r-r_\mathrm{I}|}{c}\right]e^{-i\Phi_\mathrm{I}}+\mathbf{f}(\mathbf{r}-\mathbf{r}_\mathrm{II})a_\mathrm{II}\left[t-\frac{|\mathbf{r}-\mathbf{r}_\mathrm{II}|}{c}\right]e^{-i\Phi_\mathrm{II}}. \tag{2.21}$$

Here we have introduced phase factors that we assume to be randomly distributed, thus modeling from the classical viewpoint the spontaneous emission process.

From Eq. (2.21) we find the intensity, for a fixed value of $\Delta\Phi=\Phi_\mathrm{II}-\Phi_\mathrm{I}$, to be

$$I(\mathbf{r},t)=\mathbf{E}^{(-)}(\mathbf{r},t)\mathbf{E}^{(+)}(\mathbf{r},t)=\mathbf{f}^2(\mathbf{r}-\mathbf{r}_\mathrm{I})\left|a_\mathrm{I}\left[t-\frac{|\mathbf{r}-\mathbf{r}_\mathrm{I}|}{c}\right]\right|^2+\mathbf{f}^2(\mathbf{r}-\mathbf{r}_\mathrm{II})\left|a_\mathrm{II}\left[t-\frac{|\mathbf{r}-\mathbf{r}_\mathrm{II}|}{c}\right]\right|^2$$
$$+\mathbf{f}(\mathbf{r}-\mathbf{r}_\mathrm{I})\mathbf{f}(\mathbf{r}-\mathbf{r}_\mathrm{II})\left[a_\mathrm{I}^*\left[t-\frac{|\mathbf{r}-\mathbf{r}_\mathrm{I}|}{c}\right]a_\mathrm{II}\left[t-\frac{|\mathbf{r}-\mathbf{r}_\mathrm{II}|}{c}\right]e^{-i\Delta\Phi}+\text{c.c.}\right]. \tag{2.22}$$

The average intensity is simply the sum of the intensities in the individual dipole waves,

$$\langle\langle I(\mathbf{r},t)\rangle\rangle=\mathbf{f}^2(\mathbf{r}-\mathbf{r}_\mathrm{I})\left|a_\mathrm{I}\left[t-\frac{|\mathbf{r}-\mathbf{r}_\mathrm{I}|}{c}\right]\right|^2+\mathbf{f}^2(\mathbf{r}-\mathbf{r}_\mathrm{II})\left|a_\mathrm{II}\left[t-\frac{|\mathbf{r}-\mathbf{r}_\mathrm{II}|}{c}\right]\right|^2. \tag{2.23}$$

More interesting is the intensity correlation function. Introducing the abbreviations

$$\mathbf{f}_{\nu j}=\mathbf{f}(\mathbf{r}_\nu-\mathbf{r}_j),\quad T_{\nu j}=t-\frac{|\mathbf{r}_\nu-\mathbf{r}_j|}{c}\quad(\nu=1,2;\ j=\mathrm{I},\mathrm{II}) \tag{2.24}$$

and utilizing Eq. (2.22), we find, after averaging over $\Delta\Phi$,

$$G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)\equiv\langle\langle I(\mathbf{r}_1,t)I(\mathbf{r}_2,t)\rangle\rangle$$
$$=[\mathbf{f}_{1\mathrm{I}}^2|a_\mathrm{I}(T_{1\mathrm{I}})|^2+\mathbf{f}_{1\mathrm{II}}^2|a_\mathrm{II}(T_{1\mathrm{II}})|^2][\mathbf{f}_{2\mathrm{I}}^2|a_\mathrm{I}(T_{2\mathrm{I}})|^2+\mathbf{f}_{2\mathrm{II}}^2|a_\mathrm{II}(T_{2\mathrm{II}})|^2]$$
$$+(\mathbf{f}_{1\mathrm{I}}\mathbf{f}_{1\mathrm{II}})(\mathbf{f}_{2\mathrm{II}}\mathbf{f}_{2\mathrm{I}})[a_\mathrm{I}^*(T_{1\mathrm{I}})a_\mathrm{I}(T_{2\mathrm{I}})a_\mathrm{II}^*(T_{2\mathrm{II}})a_\mathrm{II}(T_{1\mathrm{II}})+\text{c.c.}]. \tag{2.25}$$

Obviously, the final terms in square brackets describe an interference effect. According to Eqs. (2.18) and (2.24), they depend on $\mathbf{r}_1$ and $\mathbf{r}_2$ as follows:

$$\text{interference terms}=(\mathbf{f}_{1\mathrm{I}}\mathbf{f}_{1\mathrm{II}})(\mathbf{f}_{2\mathrm{II}}\mathbf{f}_{2\mathrm{I}})\tilde{a}_\mathrm{I}(T_{1\mathrm{I}})\tilde{a}_\mathrm{I}(T_{2\mathrm{I}})\tilde{a}_\mathrm{II}(T_{2\mathrm{II}})\tilde{a}_\mathrm{II}(T_{1\mathrm{II}})\exp[ik(r_{2\mathrm{I}}-r_{2\mathrm{II}}+r_{1\mathrm{II}}-r_{1\mathrm{I}})]+\text{c.c.}, \tag{2.26}$$

where the abbreviation

$$r_{\nu j}=|\mathbf{r}_\nu-\mathbf{r}_j|\quad(\nu=1,2;j=\mathrm{I},\mathrm{II}) \tag{2.27}$$

has been used and k denotes the wave number, $k=\omega/c$.

In cases of practical interest, the distance between the two emitters, as well as that between the points $\mathbf{r}_1$ and $\mathbf{r}_2$ in the receiving plane, will be small compared to the distance of $\mathbf{r}_1$ and $\mathbf{r}_2$ from the emitters. Then the f's will be nearly equal,

$$\mathbf{f}_{1\mathrm{I}}\approx\mathbf{f}_{2\mathrm{I}}\approx\mathbf{f}_{1\mathrm{II}}\approx\mathbf{f}_{2\mathrm{II}}\approx\mathbf{f}. \tag{2.28}$$

Let us assume that the emission process starts at $t=0$ in both oscillators. Since the damping constant Γ in Eq. (2.18) will be small in comparison to the oscillation frequency ω, we can neglect the variation in the damping factor $\tilde{a}$, at a given time t, i.e., we can put

$$\tilde{a}_\mathrm{I}(T_{1\mathrm{I}})\approx\tilde{a}_\mathrm{I}(T_{2\mathrm{I}})\approx\tilde{a}_\mathrm{II}(T_{1\mathrm{II}})\approx\tilde{a}_\mathrm{II}(T_{2\mathrm{II}})\approx\tilde{a}. \tag{2.29}$$

Note that $\tilde{a}$ still depends on the observation time t. Of course, the latter should be chosen such that $\tilde{a}$ comes close to its maximum value.

In the approximation described by Eqs. (2.28) and (2.29), the variation of $G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)$ is determined by the exponential in Eq. (2.26), and Eq. (2.25) reduces to the simple form

$$G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)=4\mathbf{f}^4\tilde{a}^4\{1+\tfrac{1}{2}\cos[k(r_{2\mathrm{I}}-r_{2\mathrm{II}}+r_{1\mathrm{II}}-r_{1\mathrm{I}})]\}. \tag{2.30}$$

Comparing this result with the intensity (2.22) (for fixed $\Delta\Phi$) which, under the assumptions (2.28) and (2.29), reads

$$I(\mathbf{r},t)=2\mathbf{f}^2\tilde{a}^2\{1+\cos[k(|\mathbf{r}-\mathbf{r}_{\mathrm{I}}|-|\mathbf{r}-\mathbf{r}_{\mathrm{II}}|)+\Delta\Phi]\}\ , \tag{2.31}$$

one recognizes that the relationship between $G^{(2)}$ and I is of the same kind as in the case of two plane waves with fluctuating phases considered above [see Eqs. (2.5) and (2.2)]. The results even become formally identical when we specialize to the geometrical configuration depicted in Fig. 1.

Under the assumptions $s \ll D$; $|z_1|$, $|z_2| \ll D$, the distances $r_{\nu j}$, to a good approximation, are given by

$$r_{\nu\mathrm{I}}=D\left[1+\frac{1}{2}\left[\frac{s/2-z_\nu}{D}\right]^2\right]\ ,$$
$$r_{\nu\mathrm{II}}=D\left[1+\frac{1}{2}\left[\frac{s/2+z_\nu}{D}\right]^2\right],\quad(\nu=1,2) \tag{2.32}$$

which implies

$$r_{\nu\mathrm{II}}-r_{\nu\mathrm{I}}=\frac{sz_\nu}{D}\quad(\nu=1,2)\ . \tag{2.33}$$

Hence we need only identify the ratio ks/D with Δk in the previous example [with $(\mathbf{k}_{\mathrm{II}}-\mathbf{k}_{\mathrm{I}})(\mathbf{r}_2-\mathbf{r}_1)$ replaced by $\Delta k(z_2-z_1)$ in Eq. (2.5)] to get identical results.

For later comparison with the literature, we rewrite Eq. (2.33) in a different way. First, one observes from Fig. 1 that the z direction coincides with the direction of $\mathbf{r}_{\mathrm{I}}-\mathbf{r}_{\mathrm{II}}$. Hence the right-hand side of Eq. (2.33) can be written as $D^{-1}(\mathbf{r}_{\mathrm{I}}-\mathbf{r}_{\mathrm{II}})\mathbf{r}_\nu$, where the point midway between $\mathbf{r}_{\mathrm{I}}$ and $\mathbf{r}_{\mathrm{II}}$ has been taken as the origin of the coordinate system (see Fig. 1). Now, the vector $\mathbf{r}_\nu D^{-1}$ is approximately a unit vector, in the above approximation $|z_\nu| \ll D$. It points to the position $\mathbf{r}_\nu$ where one of the detectors is placed, and hence indicates—in a naive photon picture which we shall adopt for the present—the direction in which a photon must have been emitted when it is observed at $\mathbf{r}_\nu$. (Actually, the direction of $\mathbf{r}_\nu D^{-1}$ differs a little from the "true" direction of emission given by the line that links the position of either the first or the second atom with the point $\mathbf{r}_\nu$ where the observation is made. This deviation is, however, negligibly small in our approximation.) We can thus interpret the vector $k\mathbf{r}_\nu D^{-1}$ as the wave vector of a photon $\mathbf{k}_\nu$ that arrives at $\mathbf{r}_\nu$ when emitted by either atom I or atom II. We can thus replace Eq. (2.33) by

$$k(r_{\nu\mathrm{II}}-r_{\nu\mathrm{I}})=\mathbf{k}_\nu(\mathbf{r}_{\mathrm{I}}-\mathbf{r}_{\mathrm{II}})\quad(\nu=1,2)\ , \tag{2.34}$$

which allows us to rewrite Eq. (2.30) as

$$G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)=4\mathbf{f}^4\tilde{a}^4\{1+\tfrac{1}{2}\cos[(\mathbf{k}_2-\mathbf{k}_1)(\mathbf{r}_{\mathrm{II}}-\mathbf{r}_{\mathrm{I}})]\}\ . \tag{2.35}$$

Until now, we have dealt with a common time argument t in the intensity correlation function. This restriction is, however, not essential. In fact, in the case of two time arguments t_1 and t_2, we need only make the following identification in Eq. (2.25):

$$T_{\nu j}=t_\nu-\frac{|\mathbf{r}_\nu-\mathbf{r}_j|}{c}\quad(\nu=1,2;j=\mathrm{I},\mathrm{II})\ . \tag{2.36}$$

Now, one observes from Eqs. (2.18) and (2.25) that the phase factors $\exp(i\omega t_\nu)$, $\exp(-i\omega t_\nu)$ cancel separately for $\nu=1$ and $\nu=2$. Hence Eq. (2.30) remains valid [provided, of course, that t_1 and t_2 do not differ so much as to invalidate the assumption (2.29)].

It is a simple matter, too, to generalize our result (2.30) in a different respect. Instead of two dipoles let us consider two groups of dipoles distributed around the points $\mathbf{r}_{\mathrm{I}}$ and $\mathbf{r}_{\mathrm{II}}$, respectively. The distance of any individual dipole from the corresponding center ($\mathbf{r}_{\mathrm{I}}$ or $\mathbf{r}_{\mathrm{II}}$) is, however, so small that the relevant path differences $k(|\mathbf{r}-\mathbf{r}_i|-|\mathbf{r}-\mathbf{r}_k|)$, where $\mathbf{r}$ is any point in the receiving plane and $\mathbf{r}_i,\mathbf{r}_k$ are the positions of any two oscillators, do not change noticeably when the coordinates $\mathbf{r}_i,\mathbf{r}_k$ are replaced by those of the respective centers. Using the label m (n) for dipoles in the vicinity of $\mathbf{r}_{\mathrm{I}}$ ($\mathbf{r}_{\mathrm{II}}$), we can thus write approximately

$$k(|\mathbf{r}-\mathbf{r}_m|-|\mathbf{r}-\mathbf{r}_{m'}|)=k(|\mathbf{r}-\mathbf{r}_n|-|\mathbf{r}-\mathbf{r}_{n'}|)=0\ , \tag{2.37}$$

$$k(|\mathbf{r}-\mathbf{r}_m|-|\mathbf{r}-\mathbf{r}_n|)=k(|\mathbf{r}-\mathbf{r}_{\mathrm{I}}|-|\mathbf{r}-\mathbf{r}_{\mathrm{II}}|)\ . \tag{2.38}$$

Under these conditions, Eq. (2.30) is easily extended to the present case. Let us denote the number of oscillators in the two groups by M and N, respectively. Looking at Eq. (2.25), we find the following contributions to $G^{(2)}$ (suppressing, for the moment, the common factor $\mathbf{f}^4\tilde{a}^4$):

(a) the sum of the (average) intensities of all the individual waves, squared, i.e., $(M+N)^2$;

(b) the terms similar to Eq. (2.26), describing the interference between waves emitted by any two oscillators belonging to the *same* group; in the approximation (2.37) they sum up to yield

$$M(M-1)+N(N-1)\ ;$$

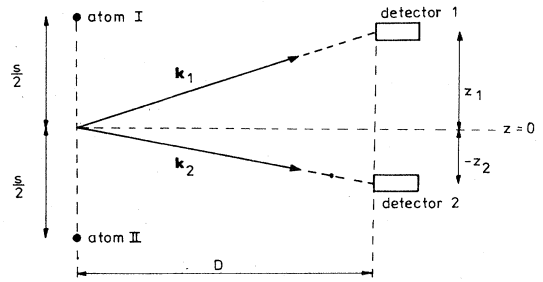

FIG. 1. Geometry for the observation of intensity correlations.

(c) the "genuine" interference terms of the type (2.26), connected with the emission of two oscillators that are members of *different* groups; by virtue of Eq. (2.38), they give us

$$2MN\cos[k(r_{2\mathrm{I}}-r_{2\mathrm{II}}+r_{1\mathrm{II}}-r_{1\mathrm{I}})] .$$

We thus arrive at the following expression for the intensity correlation function:

$$G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t) = \mathbf{f}^4\tilde{a}^4\{(M+N)^2+M(M-1)+N(N-1) +2MN\cos[k(r_{2\mathrm{I}}-r_{2\mathrm{II}}+r_{1\mathrm{II}}-r_{1\mathrm{I}})]\} . \tag{2.39}$$

Setting $M=N$ and assuming N to be large, we obtain

$$G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)\approx 6N^2\mathbf{f}^4\tilde{a}^4\{1+\tfrac{1}{3}\cos[k(r_{2\mathrm{I}}-r_{2\mathrm{II}} +r_{1\mathrm{II}}-r_{1\mathrm{I}})]\} , \tag{2.40}$$

which closely resembles our previous result (2.7) describing the interference between two plane waves with thermally fluctuating intensities. Of course, this coincidence is no accident. In fact, in our present model the two light beams originating from the two groups of oscillators, respectively, exhibit strong intensity fluctuations, being of thermal character for $N\gg 1$, as a result of the incoherent superposition of many elementary waves.

In conclusion, the main result of our analysis thus far is that interference effects become manifest in intensity correlations, even under circumstances that prevent the observation of interference patterns by conventional means.

III. INTERFERENCE BETWEEN INDEPENDENT LASER BEAMS

A. Intense beams

By virtue of its excellent monochromaticity and directivity, in conjunction with amplitude stabilization, a laser beam comes very close to what is classically described as a monochromatic traveling plane wave with fixed phase and amplitude. It should be kept in mind, however, that this holds true only for time intervals shorter than the coherence time.

Under ideal experimental conditions (fixed positions of the resonator mirrors, constant pump strength, etc.) the coherence time t_{coh} is predominantly determined by phase fluctuations: the phase in the laser field is subjected to some kind of diffusion process, and the length of the time interval over which the phase remains virtually constant, on average, defines t_{coh}. The reciprocal value of t_{coh} then gives us the bandwidth (in the sense of a fundamental lower limit), which is, in fact, extremely small. (It is about 10^{-3} Hz for typical gas laser conditions.)

In practice, however, the frequency of a single-mode laser undergoes variations, in the course of time, that are very much larger. [For instance, frequency drifts of 100 kHz/sec have been observed in a He-Ne laser by Javan, Ballik, and Bond (1962).] Those frequency variations caused mainly by mechanical instabilities of the laser resonator drastically restrict the length of the time interval T during which laser radiation can be described as a classical monochromatic wave (with fixed phase and amplitude). This critical value of T, which may be considered as an effective coherence time T_{coh}, is obviously determined by the requirement that the reciprocal of the frequency drift $\Delta\nu$ occurring during T_{coh} be equal to T_{coh}.

Fortunately, the (average) number of photons passing the beam cross section during T_{coh} is large in normal laser operation. Hence it becomes possible, by choosing the exposition time to be shorter than T_{coh}, to fulfill simultaneously the two essential requirements for observation of an interference pattern.

(i) The number of photons registered must be large, since only then will an interference structure become discernible. (From a few blackened spots one cannot infer the presence of an interference structure.)

(ii) The two laser beams that are made to interfere must have a definite phase each (during the observation), since otherwise the pattern will change its position and thus become wiped out when integrated over the observation time.

In photoelectric mixing experiments, the high intensity of the laser beams ensures that many photoelectrons are produced in the photocathode during any beat period, so that a well-defined beat note arises. In contrast to the observation of interference fringes, however, there is no need to restrict the observation time, since the above-mentioned frequency drifts give rise only to variations of the beat frequency.

Actually, it was in the form of beat notes that interference between independent laser beams was established, for the first time, by Javan, Ballik, and Bond (1962). In a beat experiment, two laser beams are aimed at the cathode of the same photomultiplier, and the beat signal is the component of the photocurrent that oscillates at the difference frequency $|\nu_{\mathrm{II}}-\nu_{\mathrm{I}}|$, where ν_{I} and ν_{II} are the frequencies of the two lasers.

Using two He-Ne lasers in cw operation, Javan, Ballik, and Bond (1962) clearly observed beat signals, and following the long-term variations of the beat frequency, they obtained detailed information on the frequency characteristics of their lasers, which was, in fact, their primary goal. Nowadays, the beat technique plays an important role in the measurement of laser frequencies, both relative (in comparison to a second laser) and absolute (with respect to a radiation source linked to the cesium time standard).

From the classical point of view, the findings of Javan, Ballik, and Bond (1962) are not surprising. Long before, it was known from broadcasting that radio waves from different (possibly very distant) emitters are able to inter-

fere with one another (to the listener's displeasure), though, of course, nobody conceived of such waves as ensembles of photons, which they nevertheless are. Further, transient interference effects were demonstrated by Hull (1949) with independent microwave beams. Optical mixing had already been observed in thermal light as early as 1955 by Forrester, Gudmundson, and Johnson (1955). (This experiment will be described in some detail in Sec. IV.A.)

The pioneering experiment establishing spatial interference between independent photons was performed by Magyar and Mandel (1963). They used two ruby lasers that emitted sequences of random spikes. The midfrequencies of these spikes proved to vary appreciably from pulse to pulse. (Not only were the optical lengths of the two ruby crystals different, but also these lengths varied due to thermal effects.) Hence, interference fringes could be observed only occasionally, namely, in those cases when the midfrequencies of two superposed pulses, ν_{I} and ν_{II}, happened to be nearly equal. But even small differences $|\nu_{\mathrm{II}}-\nu_{\mathrm{I}}|$, when they reach the order of the reciprocal of the exposition time, considerably reduce the fringe visibility. Further, the irregular nature of the pulse emission in the two lasers made the simultaneous occurrence of two spikes, certainly a prerequisite for interference to take place, an occasional event.

The authors succeeded in actually observing interference fringes with a maximum measured visibility of about 15%, by using the following technique (see Fig. 2): They directed the light from the two lasers onto the photocathode of an image tube and photographed the image produced on the output fluorescence screen. This tube was electronically gated. Normally, it was gated off by a negative bias voltage applied to the grid. Two monitor photodetectors feeding into a coincidence circuit caused the image tube to be gated on by a positive pulse only when two spikes were emitted in coincidence. By properly choosing the duration of the gating pulse, the observation time could be made significantly shorter than the spike duration, thus ensuring that the phases of the laser pulses would be constant during the time of exposition.

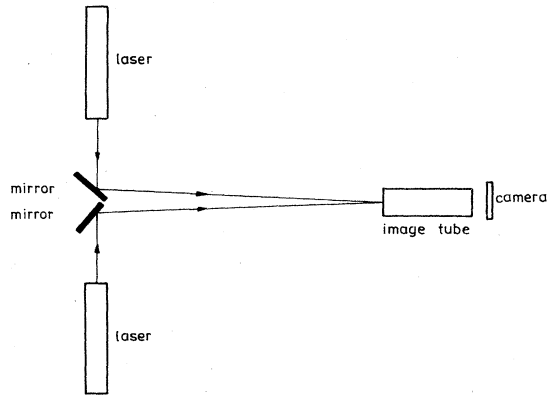

FIG. 2. Schematic diagram of the arrangement used by Magyar and Mandel (1963) to observe interference fringes produced by two ruby lasers.

The general result of the experiments by Javan, Ballik, and Bond (1962) and Magyar and Mandel (1963) was that interference between independently produced laser beams was shown to take place, in full agreement with the classical theory.

As already mentioned above, the experimenter benefits to a great extent from the high brightness of laser radiation, which ensures that a large number of photons passes the beam cross section (which is identical to the coherence area) during the coherence time. In contrast, with thermal light, as it is generated by conventional sources, the number of photons received on a coherence area during the coherence time is much less than unity. It is just this property that renders thermal light incapable of producing interference patterns. Actually, this is only a question of temperature. For source temperatures much higher than 10^5 K the photon number in question would become significantly greater than unity, and interference effects could be observed in the same manner as with lasers.

Now the interesting question arises of whether the ability of laser radiation to interfere, in the sense of interference between independent photons, is owing to its high intensity. Would it be possible to make two laser beams interfere even when they had been drastically attenuated so that only a few (or even less than unity, on average) photons were contained in the coherence volume? An answer cannot readily be given. On the one hand, certain experimental problems will arise. It has been emphasized above that the total number of registered photons must be large, anyway, for an interference pattern actually to show up. Hence it will be necessary to make the integration time much longer than the coherence time in the case of such low intensities. On the other hand, one might suspect that one cannot trust the classical predictions in these circumstances. Instead, one might expect typical quantum-mechanical effects, associated with the corpuscular nature of light, to occur.

Let us first clarify the latter issue. To this end, we have to deal with the basic features of the quantum-mechanical description of light, especially of laser radiation, and of the photoelectric detection process.

B. Quantum-mechanical description

From a formal point of view, quantization of the electromagnetic field consists in replacing the (complex) amplitudes in the expansion of the positive-frequency part of the electric field strength in terms of "modes" (usually linearly polarized monochromatic plane waves) by photon annihilation operators. Since the field within a volume over which both the amplitude and the phase of the electric field strength are constant, at any given time, represents a system with only two degrees of freedom, any segment (fixed in space) of a laser beam shorter in length

than the coherence length $cT_{\rm coh}$ may be conceived as a specific excitation of one mode of the radiation field. (Such a mode will be, in general, not simply a plane wave, but a wave with a more realistic distribution of the field amplitude over the beam cross section.) Thus we may restrict our consideration to single-mode fields. Then the positive-frequency part of the electric field strength, in operator form, reads[2]

$$E^{(+)}(\mathbf{r},t)=\mathscr{E}(\mathbf{r})q(t)\ . \tag{3.1}$$

Here we have assumed the field to be linearly polarized, $q(t)$ denotes the photon annihilation operator, and $\mathscr{E}(\mathbf{r})$ is an appropriately normalized function that describes the spatial field distribution in the respective mode.

As indicated by the argument t in Eq. (3.1), we use the Heisenberg picture. In the absence of interaction between the field and matter, $q(t)$ evolves according to

$$q(t)=qe^{-i\omega t}\ . \tag{3.2}$$

The negative-frequency part of the operator for the electric field strength is simply the Hermitian conjugate of (3.1),

$$E^{(-)}(\mathbf{r},t)=\mathscr{E}^*(\mathbf{r})q^+(t)\ , \tag{3.3}$$

where

$$q^+(t)=q^+e^{i\omega t} \tag{3.4}$$

is the photon creation operator.

It is convenient to use the Fock states $|\,n\,\rangle$ corresponding to definite photon numbers $n(=0,1,2,\ldots)$ as a basis in the Hilbert space of our system. As is well known, the operators q,q^+ act on $|\,n\,\rangle$ in the following way:

$$q\,|\,n\,\rangle=\sqrt{n}\,|\,n-1\,\rangle,\quad q^+\,|\,n\,\rangle=\sqrt{n+1}\,|\,n+1\,\rangle \qquad (n=0,1,2,\ldots)\ . \tag{3.5}$$

Now the question arises, how to describe laser radiation in the quantum-mechanical formalism. An answer is provided by the following argument: We expect that the laser field—in a segment of the beam, as specified above, and for a time interval short compared with the coherence time—comes close to what is classically described as a wave with fixed phase and amplitude [this assumption is, in fact, thoroughly substantiated by current laser theories, e.g., Haken (1970) or Paul (1969)]; hence we search for the quantum-mechanical analog of such a classical field, as the proper representation of (ideal) laser light. The desired quantum-mechanical states are easily found from the requirement that the fluctuation of the electric field strength, averaged over some light periods, be minimal, for a given mean photon number (Senitzky, 1954,1958). Expanded in terms of Fock states, these quantum-mechanical states take the form

$$|\,\alpha\,\rangle=e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|\,n\,\rangle\ , \tag{3.6}$$

where α is any complex number.

These states have been named coherent states of the field, or Glauber states. [They were originally introduced by Schrödinger (1926).]

Making use of Eqs. (3.5), one easily verifies that the following simple relations hold:

$$q\,|\,\alpha\,\rangle=\alpha\,|\,\alpha\,\rangle,\quad \langle\,\alpha\,|\,q^+=\alpha^*\langle\,\alpha\,|\ , \tag{3.7}$$

from which one observes that α has the meaning of a complex field amplitude. Since it immediately follows from Eqs. (3.7) that $\langle\,\alpha\,|\,q^+q\,|\,\alpha\,\rangle=|\,\alpha\,|^2$, the normalization is such that the square of the modulus of α equals the mean photon number.

Now contact must be made with measurement. The detection process, quite generally, utilizes the photoelectric effect. As has been shown by Glauber (1965), what a (single) detector measures is the field intensity $\langle\mathbf{E}^{(-)}(\mathbf{r},t)\mathbf{E}^{(+)}(\mathbf{r},t)\rangle$, where the angular brackets symbolize the quantum-mechanical expectation value. To speak more precisely, the counting rate R, i.e., the number of photons registered per unit time by a photodetector located at a point $\mathbf{r}$, is given by

$$R=\text{const}\times\langle\mathbf{E}^{(-)}(\mathbf{r},t)\mathbf{E}^{(+)}(\mathbf{r},t)\rangle\ . \tag{3.8}$$

Intensity measurements, however, do not exhaust the potential of photodetectors. Photodetectors also make it possible to perform coincidence counting experiments, which have proven to be a valuable means of physical investigation. We are mainly interested in the case of two detectors located at different points $\mathbf{r}_1$ and $\mathbf{r}_2$. To be counted are those events for which the first detector registers a photon and the second detector does so τ seconds later. According to Glauber (1965), the (delayed) coincidence counting rate $K(\tau)$ is proportional to the second-order correlation function for the field $G^{(2)}_{\rm qu}(\mathbf{r}_1,t;\mathbf{r}_2,t+\tau)$ [the quantum-mechanical pendant to the classical intensity correlation $\langle\langle I(\mathbf{r}_1,t)I(\mathbf{r}_2,t+\tau)\rangle\rangle$]:

$$\begin{aligned}&K(\tau)=\text{const}\times G^{(2)}_{\rm qu}(\mathbf{r}_1,t;\mathbf{r}_2,t+\tau)\ ,\\ &G^{(2)}_{\rm qu}(\mathbf{r}_1,t;\mathbf{r}_2,t+\tau)=\sum_{\rho,\sigma=1}^{3}\langle E^{(-)}_\rho(\mathbf{r}_1,t)E^{(-)}_\sigma(\mathbf{r}_2,t+\tau)E^{(+)}_\sigma(\mathbf{r}_2,t+\tau)E^{(+)}_\rho(\mathbf{r}_1,t)\rangle\ ,\end{aligned} \tag{3.9}$$

[2]Since it will always become obvious from the context whether we are dealing with quantized or classical quantities, there is no need for indicating this difference in the notation.

where the subscripts ρ and σ have been used to indicate Cartesian components. (Note that under stationary conditions the right-hand side of this equation does not depend on t.)

In both Eqs. (3.8) and (3.9), q and q^+ are arranged in normal order, and this holds true for all quantities (field correlations of any order) that can be measured with the help of photodetectors. It is due to this normal ordering that no vacuum effects (like zero-point fluctuations of the electromagnetic field) come into play. In particular, this means that it is ensured that a photodetector will never respond when the field is in the vacuum state $|0\rangle$.

The normal ordering in question has an interesting formal implication too. When calculating any field correlation for a Glauber state, we can utilize Eqs. (3.7), with the result that the quantum-mechanical expectation value is simply evaluated by replacing the operators q and q^+ by α and α^*, respectively, which makes it identical to the corresponding classical value. This complete equivalence between the quantum-mechanical and the classical description actually holds true for a much wider class of states, namely, those whose density operators ρ allow for a so-called P representation (Glauber, 1963),

$$\rho = \int P(\alpha)\,|\alpha\rangle\langle\alpha|\,d^2\alpha \tag{3.10}$$

with a non-negative function $P(\alpha)$. Here, the integration extends over the whole complex α plane. It is easy to see that the quantum-mechanical expectation value of any normally ordered product of photon creation and annihilation operators, with respect to ρ, equals the corresponding classical average over an ensemble described by the distribution function $P(\alpha)$. The extension of this equivalence theorem to the multimode case is straightforward.

It is of particular interest that thermal radiation falls into the category of fields that have a (regular) P representation (Glauber, 1963), so that the classical description proves to be perfectly correct in this relevant case.

From what has been said above it follows that we are indeed entitled to discuss interference experiments with laser beams [assuming them to be in Glauber states in times $t \lesssim T_{\rm coh}$ or, more generally, in states of the form (3.10) with $P(\alpha) \geq 0$] in purely classical terms. Hence deviations from the classical predictions—these will be specific quantum effects associated with the corpuscular nature of light—can be expected to occur only for those states of the field that do not possess a P representation, with a non-negative P function. In Sec. IV.E it will become clear that states of this type are, nevertheless, of considerable physical interest.

Let us now return to the question of what will happen when we drastically attenuate the laser beams, e.g., by means of conventional absorbers. From the classical theory one expects that such a damping process will transform a Glauber state into another Glauber state—of course, with a parameter α that has a smaller modulus. This simple picture, in fact, proves to be correct in principle (Brunner, Paul, and Richter, 1964,1965; Paul, Brunner, and Richter, 1966) for arbitrarily strong attenuation. Hence the classical description applies equally well to attenuated laser beams [cf. Paul, Brunner, and Richter (1963)], and since classical theory predicts interference to be independent of the absolute intensity level (only the relative intensities in the interfering beams are relevant), we arrive at the conclusion that interference between independent laser beams should take place, with unchanged visibility, irrespective of how strongly the beams might be attenuated (all of them in the same manner, of course). Hence this kind of interference is essentially a classical phenomenon, even at a microscopic intensity level, that is, even for photon numbers, per coherence volume, that are comparable with, or even smaller than, unity on average.

This result might, in fact, appear surprising to theoreticians, since it seemingly contradicts the famous uncertainty relation, first derived by Dirac (1927) [see also Heitler (1954)], for the phase Φ and the photon number N,

$$\Delta N \Delta\Phi \gtrsim 1\ . \tag{3.11}$$

For a Glauber state, ΔN equals $\overline{N}^{1/2}$, where $\overline{N} = |\alpha|^2$ is the mean photon number, and hence for $\overline{N} \gg 1$ this inequality is consistent with the existence of a well-defined phase. However, when $\overline{N}$ decreases, Eq. (3.11) predicts that the phase uncertainty will grow. Finally, when $\overline{N}$ goes down to values smaller than unity, the phase will become almost completely uncertain.

Since the relative phase between two interfering laser beams determines the position of the interference fringes, and the individual phases will fluctuate independently, one thus arrives at the conclusion that the interference pattern should be progressively wiped out as the intensity decreases, its visibility being completely lost for $\overline{N} \ll 1$.

Actually, this argument fails, the reason being that the phase, in the sense used by Dirac, is affected by the vacuum fluctuations of the field (in fact, the latter become dominant for $\overline{N} \lesssim 1$), whereas a photodetector does not take notice of any vacuum effect, as I mentioned above. In other words, it is not Dirac's phase that is relevant for interference experiments with lasers, but the phase of the complex number α that characterizes the Glauber state representing a laser beam even after strong attenuation.

C. Attenuated beams

While the theoretical results presented in the preceding section leave little room for doubt about the ability of low-intensity laser beams to interfere in principle, it is still a different thing to achieve such interference in practice. In fact, one might question the feasibility of such an experiment, since the physical situation seems to be no better than in the case of thermal light mentioned in Sec. III.A. Actually, when the (average) number of photons per coherence volume $\overline{N}$ is of order 1, or even smaller, no interference pattern can be produced during an exposition time shorter than $T_{\rm coh}$, simply because of the lack of pho-

tons. However, when the observation extends over a longer time interval, the interference pattern will be wiped out as a consequence of the random changes undergone by the phases in the two beams.

There is, however, a fundamental difference between attenuated laser light and thermal light. In the former case we actually start from a high-intensity field ($\overline{N} \gg 1$), and this circumstance can be utilized for appropriately controlling the conditions in the low-intensity interference experiment. In fact, what we need in order to perform the latter is information on the phases of the beams (strictly speaking, only their difference is of interest), which would enable us to choose proper periods for exposition, so that the phase difference would always have the same value in every trial. (In the language of quantum mechanics, we thus would prepare an ensemble of two-beam states of the radiation field corresponding to a fixed phase difference.) In those circumstances, the integration time could be made arbitrarily long, so that a number of photons could be registered, *in summa*, that would be sufficiently large for the desired interference pattern to show up. Actually, the required phase information can be obtained with the following procedure, as proposed by Paul, Brunner, and Richter (1965) [cf. also Paul (1966)]. Two mirrors with very low reflectivity split the intense laser beams (see Fig. 3), thus producing the two weak beams whose interference is to be observed. The (still intense) beams that have passed through the mirrors are made to interfere, and from comparison of the position of the interference pattern with that of a reference pattern, a signal is obtained for controlling the operation of a shutter. The shutter is placed before the observation screen (a photographic plate) and is opened only when the high-intensity interference pattern has the prescribed position.

Following this proposal, Radloff (1971) actually established interference between strongly attenuated laser beams originating from two independently operated He-Ne lasers. To ensure high mechanical stability of the laser resonators, he employed a cylindrical quartz block with two slits for the laser tubes and two boreholes to transmit the laser light. Attached to this block were the mirrors of both laser resonators (with inclusion, on one end, of piezoceramics). The shutter consisted of two crossed Glan-Thompson prisms with a KDP crystal between them. The control signal for the operation of the shutter was obtained from observation, with the help of a photomultiplier, of the beat note produced by the two intense beams. Whenever the beat frequency fell within the range from 3 to 70 kHz, the maxima of the photocurrent were amplified, and the rectangular pulses thus generated served to open the shutter. Since the relative phase between the two laser beams always has the same value when the ac photocurrent reaches a maximum, the shutter works in the desired way.

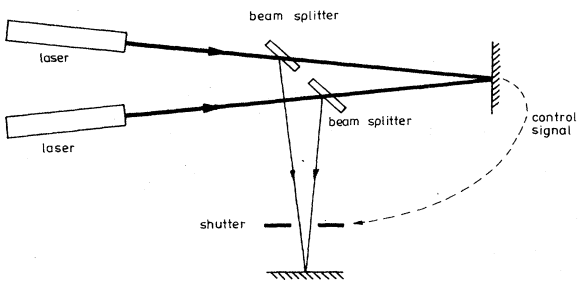

FIG. 3. Principal scheme of a setup for the observation of interference between two strongly attenuated laser beams.

In the experiment, the average incident photon flux was 10^5 photons/sec, and the shutter was opened in periods of 10^{-4}- to 10^{-5}-sec duration. Hence the number of photons passing through the shutter and falling onto the photographic plate during one period ranged from 1 to 10 photons, and was obviously too small for an interference pattern to be formed in a single trial. However, during a long exposition time (up to 30 min), all the photons reaching the plate collectively produced a clearly visible interference pattern.

I should like to add a remark that is of more theoretical interest. On principle, it is not necessary to restrict the exposition to time intervals corresponding to a given relative phase between the two beams. Instead, one might proceed as follows: One measures over an extended period the coordinates of all the spots in a photosensitive layer where photoemission processes have taken place, together with the respective times of their occurrence. These data are stored in memory; they, of course, do not indicate any interference pattern to be present. However, when a second, independent observation is made on the intense laser beams, thereby registering their relative phase as a function of time, this information can be utilized to construct an interference pattern *a posteriori* by selecting from all the accumulated data those corresponding to times in which the relative phase has a prescribed value.

It seems noteworthy that those two independent measurements need not be carried out in coincidence. A definite time delay (brought about, for example, by placing one measuring device near the two lasers, while locating the other at a large distance) does not matter, since the phases in the intense beams are certainly macroscopic quantities. Hence measuring them means only taking notice of them (without in any way disturbing them, as measurements normally would do on the microscopic level), and since they do not change during propagation in space, one can safely infer their values at a given time (at a fixed point) from delayed, as well as advanced, measurements.

D. Anticorrelations

Actually, experimental evidence of interference between attenuated laser beams was provided four years prior to Radloff's experiment by Pfleegor and Mandel (1967), though in a less direct way. They did not observe a visi-

ble interference pattern, but rather measured intensity correlations in the superposition field from two independent lasers. Since the classical description applies to laser fields even when they are strongly damped (see Sec. III.B), we can use the classical formula (2.5) to provide the theoretical background for this experiment.

Equation (2.5) describes intensity correlations in a field produced by superposition of two monochromatic plane waves with randomly (and independently) distributed phases. This is just the situation we envisage in the laser case when a continuous observation over a long period is made. As has been emphasized in Sec. III.C, one can conclude from the variation of $G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t) = \langle\langle I(\mathbf{r}_1,t)I(\mathbf{r}_2,t)\rangle\rangle$ with the difference $\mathbf{r}_2-\mathbf{r}_1$ that interference has taken place.

Now, the correlation function $G^{(2)}$ can be measured with the following procedure: One determines the number of photons n_1 and n_2, respectively, that are registered by two individual photomultipliers located at $\mathbf{r}_1$ and $\mathbf{r}_2$, during a time interval of length T; one then multiplies n_1 and n_2 and repeats the experiment many times, so that an average $\overline{n_1 n_2}$ can be formed. This average equals, up to a factor that essentially depends on the sensitivity of the detectors, the correlation function $G^{(2)}$.

Obviously, in this experiment, we must ensure that the period of exposition T is shorter than the coherence time T_{coh}, since, according to the definition of the intensity correlation (2.5), the two intensities to be multiplied correspond to fixed phases, and it is only afterwards that averaging is performed.

As has already been mentioned in Sec. II.C, the deviations $\Delta I(\mathbf{r},t)$ [see Eq. (2.13)] of the intensities from their mean values display, via their correlation (2.16), the variation of $G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)$ in a still more pronounced form. Experimentally, the quantity (2.16) is determined (up to a constant factor) by evaluating, from the data for n_1 and n_2, the average $\overline{\Delta n_1 \Delta n_2}$, where $\Delta n_\nu = n_\nu - \bar{n}_\nu$ ($\nu=1,2$). It is easily proved that the following simple relation connects $\overline{\Delta n_1 \Delta n_2}$ and $\overline{n_1 n_2}$:

$$\overline{\Delta n_1 \Delta n_2} = \overline{n_1 n_2} - \bar{n}_1 \bar{n}_2 \ . \tag{3.12}$$

Equation (2.16) predicts the correlation (3.12) to vary as $\cos[\Delta k(z_2-z_1)]$. This means it is an oscillating function of z_2-z_1 that, according to what has been said in Sec. II.C, reaches maximum when z_2-z_1 is an integer multiple of the fringe spacing, while it has its minimum values when z_2-z_1 equals an odd number of half-fringe spacings. Since the minima are actually negative, one is led to speak of anticorrelations being present in that case.

These anticorrelations have actually been measured by Pfleegor and Mandel (1967,1968) with the technique described above. To meet the above-mentioned requirement $T \ll T_{\text{coh}}$, they restricted the observation to periods of 20-μsec duration, during which the frequency difference of the two single-mode lasers fell below 50 kHz. To this end, they registered the frequency difference in the form of a beat note and passed it to a 20-μsec gate generator via a 50-kHz low-pass filter. For the observation of the interference effect a special optical arrangement was used. The light from the two lasers was made to fall upon a stack of thin glass plates, each of them about a half-fringe spacing thick. The plates were cut and arranged so that light falling upon the 1st, 3rd, 5th, etc., plate was fed to one photomultiplier, while the 2nd, 4th, 6th, etc., plate directed incident light to the other photomultiplier. The mean number of photons, $\bar{n}_1$ and $\bar{n}_2$, counted in one trial, were each about 5, at quantum efficiencies of the photomultipliers of about 7%. The number of illuminated plates was about 5.

By varying the angle between the two laser beams, the authors changed Δk, and hence the fringe spacing [see Eq. (2.10)]. In accordance with the prediction (2.16), they found the observed anticorrelations to be greatest when the thickness of the glass plates coincided with the half-fringe spacing. Moreover, their findings proved to be in reasonable quantitative agreement with the results of a detailed calculation, within the rather low statistical accuracy of the experiment.

E. Discussion

It becomes evident from the above description of the Pfleegor and Mandel experiment that such an experiment is feasible only at not too low intensities. To be more precise, the total number of photons $\overline{N}_{\text{trial}}$, on average, striking the photocathodes of both detectors in one trial must be larger than one (practically, the requirement is $\overline{N}_{\text{trial}} \gg 1$ because of the low quantum efficiencies of the photomultipliers), since otherwise the individual contributions $n_1 n_2$, from a sequence of trials, to the average $\overline{n_1 n_2}$ will vanish in the overwhelming majority of cases. Obviously, only those trials contribute to $\overline{n_1 n_2}$ in which each detector registers (at least) one photon. However, the probability for such an event to happen goes to zero as p^2, for $\overline{N}_{\text{trial}} \to 0$, where $p=(\beta/2)\overline{N}_{\text{trial}}$ (β being the detector efficiency) is the probability for an individual detector to register a photon in one trial. Actually, the interference observed in the Pfleegor and Mandel experiment took place between groups of about 70 photons, respectively.

In Radloff's experiment, in contrast, it is only required that single photons be registered from time to time. Since those events occur with probability $p'=\beta'\overline{N}_{\text{trial}}$, where β' is the detection sensitivity of the photoplate, Radloff's scheme proves to be superior to that of Pfleegor and Mandel at very low intensities. Apart from the fact that it provides an *ad oculos* demonstration of interference, it is practicable at an intensity level so low that one can speak of interference between only two photons, one from the first and one from the second laser.

While such a situation would correspond to $\overline{N}_{\text{trial}} \approx 2$, the theoretical analysis leaves no doubt that interference might be observed equally well for $\overline{N}_{\text{trial}} \lesssim 1$ or even for $\overline{N}_{\text{trial}} \ll 1$. Then it would be very unlikely that, in one trial, a photon would be present in each of the laser beams. In the majority of cases, both beams would prove to be "empty," so that nothing would happen. The events that collectively make up the ultimate interference pattern

are mainly those in which only one photon is present. If we naively attribute this photon to one of the beams (assuming it to have been emitted by either the first or the second laser), we find the situation to be paradoxical: Interference takes place not between two photons, but between one photon and "nothing."

This paradox can be resolved only in the conceptual frame of quantum mechanics. The above argument is based, in fact, on an erroneous (naive) photon picture in which photons are conceived to be something like classical particles. According to quantum mechanics, however, the photon number in a Glauber state is intrinsically indefinite; hence one is not justified in considering the number of photons in each beam (during one trial) to be a definite quantity, in the sense of classical reality. Formally, it is just this uncertainty in the photon number that brings into play the wave picture. The proper description of interference between independent photons will be as follows. What interferes with one another are waves, and when one photon is registered in Radloff's setup, one cannot say, on principle, from which laser it has come. What actually happens in that detection process is that an energy packet $h\nu$ is taken from the superposition field to which both lasers contribute equally, and hence it is only natural that this photon bears information on both laser fields that becomes manifest in the ultimate interference pattern.

IV. INTERFERENCE BETWEEN SPONTANEOUSLY EMITTED PHOTONS

A. Photoelectric mixing

The advent of the laser made interference between independent photons easy to observe, at least at high intensities. Nevertheless, it would be false to associate this type of interference only with specific states of the radiation field, generated, as is the case with laser beams, in a nonconventional way. Actually, the pioneering experiment in this field was performed before the laser era by Forrester, Gudmundson, and Johnson (1955), who succeeded in observing beats between two Zeeman components in the light from a thermal source. It is to their credit that they became discouraged neither by serious experimental difficulties nor by erroneous theoretical arguments that denied the observability of the beating phenomenon they wanted to demonstrate.

The light source employed by Forrester, Gudmundson, and Johnson (1955) was a microwave-excited electrodeless discharge in ^{202}Hg. For measurement, the line at 546.1 nm, split by an applied magnetic field into different Zeeman components, was used. The authors intended to observe optical mixing between two σ components of this spectrum. To avoid overlapping of components (their linewidth being about 10^9 Hz), the difference in their mid-frequencies was made as large as 10^{10} Hz, via Zeeman splitting. Since the two lines originate from two different sets of atoms, they are in fact incoherent.

For measurement of the beat signal, the light was focused onto a photosurface, and the ejected electrons were fed, after acceleration, into a resonant microwave cavity to which they delivered energy.

Such a measurement becomes practicable only when at least a few electrons are emitted during every period of the beat note, i.e., within 10^{-10} sec. This requires the number of photons striking the photocathode during that time to be large too. The light source was bright enough so that this condition could be fulfilled. However, the illuminated area of the photocathode covered a huge number of coherence areas in which the phases of the light waves were constant but differing, in a random manner, from area to area. As a result, the signals from all these areas added with random phases. This made the ac current from the entire cathode extremely small compared to the dc current responsible for shot noise, so that the latter provided the basic obstacle to the observation. In fact, the authors estimated the signal-to-noise ratio to be about 10^{-4}.

They overcame this difficulty by making use of a modulation technique. They modulated the intensity of the signal by passing the light from the source through a rotating half-wave plate. This caused the plane of polarization to rotate twice as rapidly as the plate. Then the light was sent though a Polaroid, which transformed the rotation of the plane of polarization, for any linearly polarized beam, into an intensity modulation. In this manner the beat signal originating from two σ components of the Zeeman-split line was modulated, while the total light intensity, and hence the shot noise, remained constant (after an initial polarization of the light source, in the absence of the magnetic field, had been compensated for). The modulated signal, after detection, was passed through a very-narrow-band (phase-selective) amplifier, before registering on the indicator.

The authors actually measured a beat signal and regarded its good agreement with the calculated signal-to-noise ratio as a confirmation of their premise that photoelectric emission is proportional to instantaneous total light intensity. From their data they drew, moreover, the important conclusion that any delay between photon absorption and electron emission must be significantly less than 10^{-10} sec, since a decay time comparable with, or longer than, the beat period would considerably decrease the signal.

B. Intensity correlations in thermal light

While Forrester, Gudmundson, and Johnson (1955) demonstrated the ability of photodetectors to follow short-term variations in light intensity, Brown and Twiss (1956a,1956b) were the first to study systematically intensity fluctuations in thermal fields, their aim being a rather

practical one. What they had in mind, and did in fact achieve, was an improvement in the measurement of stellar diameters.

Their basic idea was to replace in Michelson's stellar interferometer (Michelson, 1890,1920) the two mirrors onto which the light from the star is falling, by two large reflectors which focus the incoming light onto two similar photomultipliers (see Fig. 4). While Michelson observed the visibility of a (conventional) interference pattern, as a function of the distance between the two mirrors, Brown and Twiss (1956b) measured correlations in the photocurrents of the two detectors. To this end, the photocurrents, after having each passed a small-band amplifier, were multiplied together, and the temporal average over this product was taken as the signal.

What they thus observed was, up to a constant factor depending on the efficiencies of the detectors, the intensity correlation function $G^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)=\langle\langle I(\mathbf{r}_1,t)I(\mathbf{r}_2,t)\rangle\rangle$ (which is independent of t in stationary conditions). It should be noted that the classical description is in fact adequate in the case of thermal light, as has been mentioned in Sec. III.B.

Brown and Twiss (1956b) found the interesting result that $G^{(2)}$ exhibited a maximum at $\mathbf{r}_1=\mathbf{r}_2$. With growing distance between the reflectors $|\mathbf{r}_2-\mathbf{r}_1|$, $G^{(2)}$ decreased until a definite constant level was attained. This behavior of the intensity correlation function closely parallels the decrease of fringe visibility, as observed in Michelson's interferometer. What can be measured in both cases is a critical distance $|\mathbf{r}_2-\mathbf{r}_1|_{\text{crit}}$, given by either the width of the maximum of $G^{(2)}$ or the mirror distance for which the interference pattern becomes invisible. This quantity equals the transverse coherence length of stellar light on the Earth's surface, from which the (seeming) stellar diameter can easily be inferred. However, the stellar intensity interferometer, as it has been called by Brown and Twiss (1956b), has two important advantages: Atmospheric scintillations do not disturb the measurement, and the reflector distance can be made very large, since no mechanical connection between the reflectors is required.

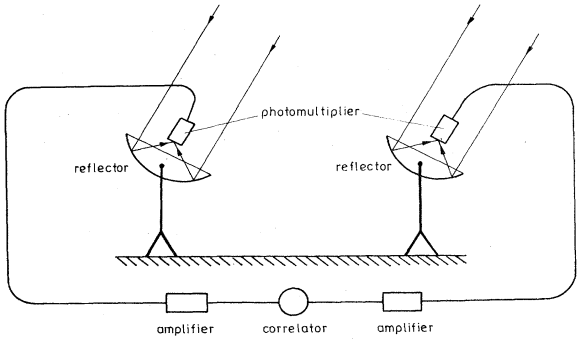

FIG. 4. Schematic diagram of the stellar interferometer developed by Brown and Twiss (1956b).

We are interested not so much in the practical aspects of the stellar intensity interferometer as in the conclusions that can be drawn from the observations on the characteristics of thermal light. From the classical point of view, the experimental results are easily explained in the following picture: The atoms on the star's surface emit, independently from each other and with random phases, elementary wave trains whose superposition (on Earth) is the subject of investigation. This superposition field undergoes fluctuations in both its phase and its amplitude. At a given time, both quantities are approximately constant over a coherence area; hence the detector outputs are correlated when their positions fall in the same area. Otherwise the photocurrents fluctuate independently. (In practice, the transverse coherence length is a rather large quantity, ranging from meters to hundreds of meters and more.)

Since superposition of a large number of individual waves is certainly an interference phenomenon, the effect measured by Brown and Twiss (1956b) can actually be regarded as an indication of interference having taken place between independently emitted elementary wave trains, i.e., photons.

However, things are hard to understand in a naive photon picture. The paradoxical character of the situation becomes still more obvious when the measurement is performed with photodetectors that count individual photons, the signal now being the coincidence counting rate. The maximum exhibited by the latter for $\mathbf{r}_1=\mathbf{r}_2$ then leads us to ask the following question. When photons are emitted independently by various atoms on the star's surface, what could bring about a tendency for any two of them to arrive at a short (transverse) distance with an enhanced probability? The origin of such a correlation remains in fact obscure when we tacitly assume that what we count as a photon can be identified with what is emitted by a particular (though unknown) atom on the star's surface. Actually, this assumption cannot be upheld; photons are not individuals whose "course of life" may be followed, at least on principle, from "birth" to "death." As has been pointed out already at the end of Sec. III.E, what is registered as a photon is an energy packet $h\nu$ taken from the superposition field, and this explains why it is possible to get information on the extension of the star's surface by counting single photons.

To summarize, we can say that the development of photoelectric detectors made it possible to measure the intensity fluctuations occurring in thermal light—not only in the form of the spatial intensity correlations considered thus far, but also in the form of correlations between the arrival times of successive photons, as was first done by Rebka and Pound (1957)—thus essentially confirming the classical ideas about the statistical features of thermal light.

While in those experiments a huge number of elementary emitters was involved, in the following we shall discuss, from the theoretical point of view, the simplest and hence most instructive case, namely, that of precisely two emitting atoms.

C. The two-atom case

We have already calculated the intensity correlation for two Hertzian oscillators in Sec. II.C. The classical description, however, turns out to fail when the emitters are identified with excited atoms. In fact, the quantum-mechanical state of the field emitted by one or two atoms cannot be described by a density operator that has a (regular) P representation. Hence the equivalence between classical and quantum-mechanical descriptions stated in Sec. III.B, no longer exists, and the two theories must be expected to yield different predictions. The discrepancy is, in fact, significant, as will become obvious from the subsequent analysis.

Let us evaluate the quantum-mechanical expression for the intensity correlation function. Our starting point is again Eq. (2.19). "Translating" it into the quantum-mechanical language means to interpret the amplitude of the oscillator as an appropriate operator (which, in turn, implies the operator character of $\mathbf{E}^{(+)}$ and $\mathbf{E}^{(-)}$). We assume the operators to be time dependent, i.e., we use the Heisenberg picture.

We idealize the atom by a two-level system. Then the operator a is a lowering operator: it brings the atom from its upper level $|2\rangle$ to its lower level $|1\rangle$,

$$a(0)|2\rangle=|1\rangle\ . \tag{4.1}$$

Correspondingly, the Hermitian conjugate of a, $a^\dagger$, is a raising operator describing the reverse transition,

$$a^\dagger(0)|1\rangle=|2\rangle\ . \tag{4.2}$$

It is well known that the operators $a,a^\dagger$ obey the anticommutation relations

$$a^2=0,\quad a^{\dagger 2}=0,\quad aa^\dagger+a^\dagger a=1\ . \tag{4.3}$$

Instead of the classical equation (2.21) we now have to write

$$\mathbf{E}^{(+)}(\mathbf{r},t)=\mathbf{f}(\mathbf{r}-\mathbf{r}_{\mathrm{I}})a_{\mathrm{I}}\left[t-\frac{|\mathbf{r}-\mathbf{r}_{\mathrm{I}}|}{c}\right]+\mathbf{f}(\mathbf{r}-\mathbf{r}_{\mathrm{II}})a_{\mathrm{II}}\left[t-\frac{|\mathbf{r}-\mathbf{r}_{\mathrm{II}}|}{c}\right], \tag{4.4}$$

where the lowering operators a_{I} and a_{II} commute. It should be noted that the classical factors $\exp(-i\Phi_{\mathrm{I}})$ and $\exp(-i\Phi_{\mathrm{II}})$ in Eq. (2.21) are already comprised in the lowering operators. In fact, it immediately follows from Eq. (4.1) that the expectation values for a_{I} and a_{II} vanish in the initial state, where both atoms are excited. Later on we shall show that this remains so during the whole evolution [see Eq. (4.6)]. This circumstance indicates that the phases of the individual dipole oscillations are randomly distributed, which, in turn, makes the expectation values for $\mathbf{E}^{(-)}$ and $\mathbf{E}^{(+)}$ vanish.

Together with its Hermitian conjugate, Eq. (4.4) forms the basis for a quantum-mechanical calculation of intensity correlations. Thus the quantum-mechanical treatment parallels the classical one presented in Sec. II.C. As a result of the aforementioned vanishing of $\langle\mathbf{E}^{(-)}\rangle$ and $\langle\mathbf{E}^{(+)}\rangle$, the mean intensity is constant in space. Hence the presence of interference effects can be inferred from higher-order field correlations only. As in Sec. II.C, we shall study intensity correlations that are described by the second-order correlation function

$$G^{(2)}_{\mathrm{qu}}(\mathbf{r}_1,t;\mathbf{r}_2,t)=\sum_{\rho,\sigma=1}^{3}\langle E^{(-)}_\rho(\mathbf{r}_1,t)E^{(-)}_\sigma(\mathbf{r}_2,t)E^{(+)}_\sigma(\mathbf{r}_2,t)E^{(+)}_\rho(\mathbf{r}_1,t)\rangle\ . \tag{4.5}$$

Because of the normal ordering of the field operators in this expression, we have to deal with expectation values for normally ordered products of (altogether four) lowering and raising operators (the latter standing to the left of the former).

Generally, we assume that the atoms emit independently from one another. This is an approximation, since they are coupled to the same radiation field. However, their mutual influence can be neglected, provided they are separated by a distance that is large compared to the wavelength of the radiation. [In the opposite case, our assumption is, in fact, not justified. Near-zone effects have to be properly taken into account which, however, do not affect the equal-time correlation function (4.5), as has been shown by Th. Richter (1979). See also Sec. IV.F.] The approximation in question enables us to factorize products of operators associated with different atoms.

Let us first consider expectation values that vanish in the classical description. It is readily seen that they do the same in the quantum-mechanical formalism.

We have already stated that the expectation values for a and $a^\dagger$ vanish at $t=0$. This also holds true during the emission process. If it did not, the atomic dipole moment [represented by the operator $d(a+a^\dagger)$] would acquire a finite value, thus indicating some preference for certain values of the dipole phase, and this would be in contrast to the intrinsically random nature of the spontaneous emission process. A formal argument is based on the equation of motion for $\langle a\rangle$ which follows from a quantum-mechanical analysis that takes proper account of radiation damping [see Mollow and Miller (1969); cf. also Mollow (1969)]:

$$\frac{d}{dt}\langle a\rangle=(-i\omega_{21}-\Gamma/2)\langle a\rangle\ . \tag{4.6}$$

Here ω_{21} is the level distance, in units of $\hbar$, and Γ the reciprocal lifetime of the upper level with respect to spontaneous emission. (The lower level is assumed to be the ground state.) Obviously, Eq. (4.6) confirms our assertion

$$\langle a(t')\rangle=\langle a^\dagger(t')\rangle=0\ \text{ for } t'\geq 0\ . \tag{4.7}$$

Making use of the above-mentioned factorizibility, one immediately concludes from Eq. (4.7) that any expectation value to which one of the atoms contributes only through one operator, either a or $a^\dagger$, vanishes too, e.g.,

$$\langle a_{\rm I}^{\dagger}(t_1)a_{\rm I}^{\dagger}(t_2)a_{\rm I}(t_3)a_{\rm II}(t_4)\rangle = \langle a_{\rm I}^{\dagger}(t_1)a_{\rm I}^{\dagger}(t_2)a_{\rm I}(t_3)\rangle\langle a_{\rm II}(t_4)\rangle = 0 . \quad (4.8)$$

Both Eqs. (4.7) and (4.8) are in perfect agreement with the classical description.

A similar argument leads to the result

$$\langle a_{\rm I}^{\dagger}(t_1)a_{\rm I}^{\dagger}(t_2)a_{\rm II}(t_3)a_{\rm II}(t_4)\rangle = \langle a_{\rm I}^{\dagger}(t_1)a_{\rm I}^{\dagger}(t_2)\rangle\langle a_{\rm II}(t_3)a_{\rm II}(t_4)\rangle = 0 . \quad (4.9)$$

In fact, terms of the type $\langle a^{\dagger}(t)a^{\dagger}(t')\rangle$ or $\langle a(t)a(t')\rangle$ are equal to zero. For $t=t'$, this is a direct consequence of the commutation relations (4.3), and for $t\neq t'$ it can be readily proved with the help of the so-called quantum regression theorem, as will be shown below. Obviously, the quantum-mechanical prediction (4.9) agrees with the corresponding classical one.

The relevant contributions to the intensity correlation function (4.5) are thus of the type [cf. Eq. (2.25)]

(i) $\langle a_{\rm I}^{\dagger}(t_1)a_{\rm II}^{\dagger}(t_2)a_{\rm II}(t_3)a_{\rm I}(t_4)\rangle$

or

(ii) $\langle a_{\rm I}^{\dagger}(t_1)a_{\rm I}^{\dagger}(t_2)a_{\rm I}(t_2)a_{\rm I}(t_1)\rangle$.

Terms of the form (i) can also be factorized,

$$\langle a_{\rm I}^{\dagger}(t_1)a_{\rm II}^{\dagger}(t_2)a_{\rm II}(t_3)a_{\rm I}(t_4)\rangle = \langle a_{\rm I}^{\dagger}(t_1)a_{\rm I}(t_4)\rangle\langle a_{\rm II}^{\dagger}(t_2)a_{\rm II}(t_3)\rangle , \quad (4.10)$$

in accordance with the classical description.

The equivalence between the classical and the quantum-mechanical theory, however, breaks down in the case of terms like (ii). In fact, the corresponding classical expression reads $|a_{\rm I}(t_1)|^2|a_{\rm I}(t_2)|^2$, which is of the same order of magnitude as the remaining terms in Eq. (2.25), while quantum theory predicts the expectation value (ii) to vanish. For $t_1=t_2$, this follows immediately from the anticommutation relations (4.3), and for $t_1\neq t_2$ a proof can be given that utilizes the quantum regression theorem (see below).

Apart from any formal argument, however, the vanishing of the expectation value (ii) is readily understood on general physical grounds. Actually, it is simply dictated by the energy conservation law: The term (ii) determines the intensity correlation function for the field emitted by just one atom, and since the intensity correlation gives us, apart from a factor, the probability of finding two photons, normally each at a different position, a nonvanishing value of (ii) would indicate the possibility of detecting two photons when only one is emitted.

It should be emphasized that the failure of the classical description in the present case is not owing to its inability to account for photons, in the sense of wave packets that contain the energy $h\nu$, but rather its inability to describe properly the photoelectric effect, which is intimately connected with the corpuscular aspect of light. In fact, the classical theory of photoelectric detection is faced with a nonresolvable problem: If it correctly accounted for the energy conservation law, no photodetector placed in the far field could respond at all to a dipole field with total energy $h\nu$, due to the lack of available energy resulting from the "dilution" of the electromagnetic energy in the course of propagation. The classical theory circumvents this difficulty, disregarding the energy conservation law, by the assumption that the probability for a detector to count a photon is proportional to the instantaneous intensity. This approach is certainly justified when the energy of the field is large compared to $h\nu$. Moreover, it also proves to be correct for coherent waves (i.e., waves with definite phases and amplitudes)—or, more generally, for fields that can be described by statistical mixtures of coherent waves in the sense of Glauber's P representation—even at (arbitrarily) low intensities, since the classical predictions are identical to the quantum-mechanical ones in this case, as has been pointed out in Sec. III.B.

Quantum mechanics actually explains why there is no conflict with the energy conservation law: The number of photons in a coherent state, in the quantum-mechanical description, is uncertain; even at arbitrarily low intensities there is a nonvanishing probability of finding one, two, or even more photons [according to Eq. (3.6), this probability follows a Poisson distribution], so that, in principle, the available energy is unlimited.

However, the classical approach necessarily fails to give results that satisfy the energy conservation law when the field under consideration contains a definite (small) number of photons that is less than the number of detectors used for observation. Then the classical theory predicts that events in which all the detectors respond to the field (i.e., register a photon each) will occur with a nonzero probability, which is certainly in contrast to the energy conservation law, and hence erroneous.

It is just such a situation that we are dealing with in the discussion of the expectation value (ii). Therefore the distinct difference between this value and its classical analog is of fundamental importance. It reflects the inability of the classical theory, based entirely on the wave concept, to account properly for the particle aspect of light.

Since the term (ii) (and the corresponding term with the subscript I replaced by II) enters the expression for the intensity correlation function in the case of two emitting atoms, the classical and the quantum theory will differ in the prediction of this correlation too. This discrepancy will then show up as a specific quantum effect.

What we still have to do is to evaluate expectation values of the type $\langle a_{\rm I}^{\dagger}(t)a_{\rm I}(t')\rangle$ [see Eq. (4.10)]. To this end, we make use of the quantum regression theorem proved by Lax (1968) for Markovian systems, which enables us to express any two-time correlation function through single-time expectation values. Its general formulation is as follows. If M is a member (or a linear combination) of a complete set of Markovian system operators M_μ, then the time evolution of the expectation value of M can be written as

$$\langle M(t)\rangle = \sum_\mu \beta_\mu(t)\langle M_\mu(0)\rangle \quad (t\geq 0) , \quad (4.11)$$

and the mean of a two-time operator $L(t)M(t+\tau)N(t)$, where L and N are any system operators, is given by

$$\langle L(t)M(t+\tau)N(t)\rangle=\sum_{\mu}\beta_{\mu}(\tau)\langle L(t)M_{\mu}(t)N(t)\rangle \quad (\tau\geq 0)\ . \tag{4.12}$$

In the present case, we start from Eq. (4.6), which is readily integrated to yield

$$\langle a(t)\rangle=\exp\left[-\left[i\omega_{21}+\frac{\Gamma}{2}\right]t\right]\langle a(0)\rangle\ . \tag{4.13}$$

Applying the regression theorem, we then find

$$\langle a^{\dagger}(t)a(t+\tau)\rangle=\exp\left[-\left[i\omega_{21}+\frac{\Gamma}{2}\right]\tau\right]\langle a^{\dagger}(t)a(t)\rangle \quad (\tau\geq 0)\ , \tag{4.14}$$

and in a similar way we obtain

$$\langle a^{\dagger}(t+\tau)a(t)\rangle=\exp\left[\left[i\omega_{21}+\frac{\Gamma}{2}\right]\tau\right]\langle a^{\dagger}(t)a(t)\rangle \quad (\tau\geq 0)\ . \tag{4.15}$$

It is interesting to note that these results are identical to the corresponding classical ones. In fact, according to Eqs. (2.21) and (2.18), the classical analog of the correlation (4.15), for instance, reads

$$\langle\langle[a(t+\tau)e^{-i\Phi}]^{*}a(t)e^{-i\Phi}\rangle\rangle=a^{*}(t+\tau)a(t)=\exp\left[\left[i\omega_{21}+\frac{\Gamma}{2}\right]\tau\right]|a(t)|^{2}\ , \tag{4.16}$$

and the values of $|a(t)|^2$ and $\langle a^{\dagger}(t)a(t)\rangle$ are identical: they are both equal to $\exp(-\Gamma t)$, as follows from Eq. (2.18) and the quantum-mechanical equation of motion

$$\frac{d}{dt}\langle a^{\dagger}a\rangle=-\Gamma\langle a^{\dagger}a\rangle \tag{4.17}$$

for the initial condition $\langle a^{\dagger}(0)a(0)\rangle=1$. [Since the operator $a^{\dagger}a$ characterizes the occupation of the upper atomic level, Eq. (4.17) describes the exponential decay of the excited atomic state. It is derived in the same manner as Eq. (4.6).]

Finally, we learn from the regression theorem that two-time correlation functions $\langle a(t+\tau)a(t)\rangle$, $\langle a^{\dagger}(t+\tau)a^{\dagger}(t)\rangle$ differ from their values at $\tau=0$ only by a factor, and hence vanish by virtue of Eqs. (4.3). The same argument also applies to the expectation value (ii). [Here, the proof starts from Eq. (4.17).]

Summing up our results, we can state that the contributions to the intensity correlation function are identical in the classical and the quantum-mechanical description, with one important exception: The terms of the form (ii) vanish in the quantum-mechanical formalism, whereas they are of the same order of magnitude as the remaining contributions in the classical theory.

Hence the quantum-mechanical expression for the intensity correlation is immediately obtained from the classical formula (2.25) by dropping the form (ii) terms. We thus arrive at the result

$$G^{(2)}_{\rm qu}(\mathbf{r}_1,t;\mathbf{r}_2,t)=\mathbf{f}^2_{1\rm II}\mathbf{f}^2_{2\rm II}\,|\,a_{\rm I}(T_{1\rm II})a_{\rm II}(T_{2\rm II})\,|^2+\mathbf{f}^2_{2\rm I}\mathbf{f}^2_{1\rm II}\,|\,a_{\rm I}(T_{2\rm I})a_{\rm II}(T_{1\rm II})\,|^2 +(\mathbf{f}_{1\rm I}\mathbf{f}_{1\rm II})(\mathbf{f}_{2\rm II}\mathbf{f}_{2\rm I})[a^{*}_{\rm I}(T_{1\rm I})a_{\rm I}(T_{2\rm I})a^{*}_{\rm II}(T_{2\rm II})a_{\rm II}(T_{1\rm II})+\text{c.c.}]\ , \tag{4.18}$$

where the a's are the classical variables (2.18).

In the approximations (2.28) and (2.29), we have to subtract $2\mathbf{f}^4\tilde{a}^4$ from the classical expression (2.30), the result being

$$G^{(2)}_{\rm qu}(\mathbf{r}_1,t;\mathbf{r}_2,t)=2\mathbf{f}^4\tilde{a}^4\{1+\cos[k(r_{2\rm I}-r_{2\rm II}+r_{1\rm II}-r_{1\rm I})]\}\ . \tag{4.19}$$

Specializing to the configuration represented by Fig. 1, we have to replace Eq. (2.35) by

$$G^{(2)}_{\rm qu}(\mathbf{r}_1,t;\mathbf{r}_2,t)=2\mathbf{f}^4\tilde{a}^4\{1+\cos[(\mathbf{k}_2-\mathbf{k}_1)(\mathbf{r}_{\rm II}-\mathbf{r}_{\rm I})]\}\ . \tag{4.20}$$

The difference between Eqs. (2.30) and (4.19), or between Eqs. (2.35) and (4.20), is, in fact, of great physical relevance. While the possible minimum value of $G^{(2)}$ is one-half of the average value in the classical description, it is zero in the quantum-mechanical formalism.

Physically, the quantum-mechanical result means that two detectors can be placed at positions such that they will never *both* respond. In the configuration illustrated by Fig. 1 this happens whenever the distance between the two detectors equals an odd half-number of fringe spacings. It becomes obvious in the above derivation of the quantum-mechanical expression for $G^{(2)}$ that this effect is quantum-mechanical in nature, having no classical analogy, the reason being the inability of classical theory to cope with the particle aspect of light.

In any case, the presence of intensity correlations is testimony that interference has taken place. It is interesting to see, however, that quantum mechanics predicts those correlations to be significantly stronger than in classical theory. One might say that the quantum nature of spontaneous emission produces, in the specific conditions presently under consideration, a higher degree of "structural order" than one would expect from the classical theory.

D. Some generalizations

As in Sec. II.C we still generalize our result (4.19) to the case in which the two sources each contain a larger number of excited atoms, N and M, respectively. Repeating the argumentation that led us to Eq. (2.39), we have to observe that single-atom contributions like (ii) vanish in the quantum-mechanical description. This means that we must subtract the term $(M+N)\mathbf{f}^4\tilde{a}^4$ in the classical formula (2.39), which gives us

$$G_{\rm qu}^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)=2\mathbf{f}^4\tilde{a}^4\{M(M-1)+N(N-1)+MN+MN\cos[k(r_{2\rm I}-r_{2\rm II}+r_{1\rm II}-r_{1\rm I})]\}\ . \tag{4.21}$$

A formula of this type was first derived by Mandel (1983).

For M and $N\gg 1$, the quantum-mechanical correction is, in fact, negligibly small, so that the classical and the quantum-mechanical description practically agree, which is not surprising, of course. In particular, Eq. (2.40) holds in both theories in the case $M=N(\gg 1)$.

More interesting is the situation in which the numbers of atoms, M and N, are not fixed but rather (independently) fluctuating. We then have to average Eq. (4.21) over those numbers. Assuming them to be Poisson distributed, $\overline{M(M-1)}=\overline{M}^2$, $\overline{N(N-1)}=\overline{N}^2$, we find the simple result

$$G_{\rm qu}^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)=2\mathbf{f}^4\tilde{a}^4\{\overline{M}^2+\overline{N}^2+\overline{M}\,\overline{N}+\overline{M}\,\overline{N}\cos[k(\mathbf{r}_{2\rm I}-r_{2\rm II}+r_{1\rm II}-r_{1\rm I})]\}\ . \tag{4.22}$$

In the special case $\overline{M}=\overline{N}$, we again arrive at Eq. (2.40) (with N replaced by $\overline{N}$), characteristic of thermal radiation, which now, however, holds exactly, without the assumption that $\overline{M}$ and $\overline{N}$ are large (Mandel, 1983). Hence the above-mentioned nonclassical effect that is present in the case of precisely two atoms disappears, as a consequence of fluctuations in the numbers of atoms, even when $\overline{N}$ and $\overline{M}$ are small compared to unity. Obviously, this is related to the fact that there is a nonvanishing probability that one and the same source will emit two, or even more, photons so that two photons being registered by the two detectors, in some cases, may originate from one and the same source, a feature inherent, quite generally, in the classical description.

It should be noted, however, that the classical formula differs drastically from the quantum-mechanical one for $\overline{M}\ll 1$, $\overline{N}\ll 1$, since then the extra term $(\overline{M}+\overline{N})\mathbf{f}^4\tilde{a}^4$ in the classical expression clearly dominates the interference term, which is proportional to $\overline{M}\overline{N}$. Hence the intensity correlation function is virtually constant, giving no indication of interference taking place.

Thus far we have assumed that the atoms idealized as two-level systems become initially fully excited by some pumping mechanism. Modern laser technology, however, makes it possible to realize more general initial conditions. With the help of intense short laser pulses, the atoms can be prepared in a coherent superposition of the form

$$|\psi\rangle=c_1e^{i\Phi}|1\rangle+c_2|2\rangle\ , \tag{4.23}$$

where the coefficients c_1 and c_2, assumed real, satisfy the normalization condition $c_1^2+c_2^2=1$.

Our former considerations (corresponding to $c_1=0$) are easily extended to the present case (Mandel, 1983). The new feature is that the expectation values for a and $a^\dagger$ no longer vanish, which means that a dipole moment has been induced on the atoms by the pumping pulse. For simplicity, we assume that the coefficients c_1 and c_2 are the same for all atoms; however, we allow the phase angle Φ to differ for the two groups of atoms that constitute the two sources.

From Eq. (4.23) we easily find the only nonvanishing expectation values (for normally ordered operators) to be

$$\langle\psi|a|\psi\rangle=c_1c_2e^{-i\Phi},\ \langle\psi|a^\dagger|\psi\rangle=c_1c_2e^{i\Phi}\ ,\qquad \langle\psi|a^\dagger a|\psi\rangle=c_2^2\ . \tag{4.24}$$

In the two-atom case, the dipole oscillation in question actually has no influence on the intensity correlation function $G_{\rm qu}^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)$ since $\langle a_{\rm II}(t_4)\rangle$, for instance, is multiplied by the factor $\langle a_{\rm I}^\dagger(t_1)a_{\rm I}^\dagger(t_2)a_{\rm I}(t_3)\rangle$ [see Eq. (4.8)], which vanishes by virtue of the anticommutation relations (4.3) in conjunction with the quantum regression theorem. Hence $G_{\rm qu}^{(2)}(\mathbf{r}_1,t;\mathbf{r}_2,t)$ is essentially the same as in the former case $c_1=0$, $c_2=1$, the only difference being the appearance of a common factor c_2^2.

It should be noted that the situation is different when more than two atoms come into play. Then terms of the form

$$\langle a_i^\dagger(t_1)a_i(t_2)\rangle\langle a_j^\dagger(t_3)\rangle\langle a_k(t_4)\rangle$$

or

$$\langle a_i^\dagger(t_1)\rangle\langle a_j^\dagger(t_2)\rangle\langle a_k(t_3)\rangle\langle a_l(t_4)\rangle$$

(with all atomic labels $i,j,\ldots$ being different) will contribute to the intensity correlation function, thus making it dependent on the difference of the phases $\Phi_{\rm I}$ and $\Phi_{\rm II}$ corresponding to the two sources. [For details see Mandel (1983).]

The presence of induced dipole moments has an important implication for the intensity itself. In fact, the intensity becomes modulated in space, thus directly displaying an interference pattern. Utilizing Eq. (4.4), we can write the intensity in the form [cf. also Eq. (2.22)]

$$G_{\rm qu}^{(1)}(\mathbf{r},t)\equiv\langle\mathbf{E}^{(-)}(\mathbf{r},t)\mathbf{E}^{(+)}(\mathbf{r},t)\rangle=\mathbf{f}^2(\mathbf{r}-\mathbf{r}_{\rm I})\langle a_{\rm I}^\dagger(T_{\rm I})a_{\rm I}(T_{\rm I})\rangle+\mathbf{f}^2(\mathbf{r}-\mathbf{r}_{\rm II})\langle a_{\rm II}^\dagger(T_{\rm II})a_{\rm II}(T_{\rm II})\rangle+\mathbf{f}(\mathbf{r}-\mathbf{r}_{\rm I})\mathbf{f}(\mathbf{r}-\mathbf{r}_{\rm II})[\langle a_{\rm I}^\dagger(T_{\rm I})\rangle\langle a_{\rm II}(T_{\rm II})\rangle+\text{c.c.}]\ , \tag{4.25}$$

where the abbreviations

$$T_{\rm I}=t-\frac{|\mathbf{r}-\mathbf{r}_{\rm I}|}{c},\quad T_{\rm II}=t-\frac{|\mathbf{r}-\mathbf{r}_{\rm II}|}{c} \tag{4.26}$$

have been introduced.

In the approximation described by Eqs. (2.28) and (2.29), Eq. (4.25) reduces to

$$G^{(1)}_{\rm qu}(\mathbf{r},t)=2\mathbf{f}^2\tilde{a}^2(T_{\rm I})c_2^2[1+c_1^2\cos(k\,|\mathbf{r}-\mathbf{r}_{\rm I}|-k\,|\mathbf{r}-\mathbf{r}_{\rm II}|+\Phi_{\rm II}-\Phi_{\rm I})]\,, \tag{4.27}$$

where use has been made of Eqs. (4.24), (4.25), (4.17), (4.6), and (2.18).

The visibility of the interference pattern (4.27) is c_1^2 and thus is generally worse than in the corresponding classical case, where it equals unity. It approaches the classical value for $c_1\to 1$. However, then c_2 goes to zero so that the intensity, and hence the detection probability, becomes vanishingly small.

E. Fields with fixed photon numbers

In Secs. IV.C and IV.D we studied intensity correlations that are present in the field produced by spontaneously radiating atoms, taking explicitly into account the emission process. Since emission has actually finished when the observation is made at a wide distance from the sources, free fields traveling in space can be used for the description equally well. The essential point is that these fields, when emitted from a definite number of atoms, will contain the same fixed number of photons. I shall show in the following how the same intensity correlations can be recovered in the free-field formalism.

Let us first consider the simple case of two linearly polarized traveling plane waves (labeled I and II) with equal frequencies but slightly different propagation directions. These waves can be identified with specific modes of the radiation field. Since the measurable quantities like intensity and intensity correlations are represented by normally ordered products of $\mathbf{E}^{(-)}$ and $\mathbf{E}^{(+)}$, the remaining modes, being in their vacuum states, give no contributions, so that only the nonempty modes need to be taken into account.

We are thus able to write

$$\mathbf{E}^{(+)}(\mathbf{r},t)=\mathbf{e}_{\rm I}(\mathbf{r},t)q_{\rm I}+\mathbf{e}_{\rm II}(\mathbf{r},t)q_{\rm II}\,. \tag{4.28}$$

Here, $q_{\rm I},q_{\rm II}$ are the familiar photon annihilation operators taken at $t=0$, and

$$\mathbf{e}_j(\mathbf{r},t)=i\left[\frac{2\pi\hbar\omega_j}{V}\right]^{1/2}\exp[i(\mathbf{k}_j\mathbf{r}-\omega_j t)]\mathbf{p}_j \qquad (j={\rm I,II}) \tag{4.29}$$

is the classical field strength, in proper normalization, V being the mode volume and $\mathbf{p}_j$ a unit vector indicating the polarization direction.

Let us assume that the photon numbers in the two fields, m and n, are sharp. Then the expectation values for q, $q^\dagger$, $q^{\dagger 2}q$, and $q^\dagger q^2$ vanish, and we find from Eq. (4.28)

$$G^{(1)}_{\rm qu}(\mathbf{r},t)=\langle\mathbf{E}^{(-)}(\mathbf{r},t)\mathbf{E}^{(+)}(\mathbf{r},t)\rangle=|\,\mathbf{e}_{\rm I}(\mathbf{r},t)\,|^2\langle q_{\rm I}^\dagger q_{\rm I}\rangle+|\,\mathbf{e}_{\rm II}(\mathbf{r},t)\,|^2\langle q_{\rm II}^\dagger q_{\rm II}\rangle\,, \tag{4.30}$$

$$\begin{aligned}G^{(2)}_{\rm qu}(\mathbf{r}_1,t;\mathbf{r}_2,t)&=\sum_{\rho,\sigma=1}^{3}\langle E_\rho^{(-)}(\mathbf{r}_1,t)E_\sigma^{(-)}(\mathbf{r}_2,t)E_\sigma^{(+)}(\mathbf{r}_2,t)E_\rho^{(+)}(\mathbf{r}_1,t)\rangle\\&=|\,\mathbf{e}_{\rm I}(\mathbf{r}_1,t)\,|^2|\,\mathbf{e}_{\rm I}(\mathbf{r}_2,t)\,|^2\langle q_{\rm I}^{\dagger 2}q_{\rm I}^2\rangle+|\,\mathbf{e}_{\rm II}(\mathbf{r}_1,t)\,|^2|\,\mathbf{e}_{\rm II}(r_2,t)\,|^2\langle q_{\rm II}^{\dagger 2}q_{\rm II}^2\rangle\\&\quad+|\,\mathbf{e}_{\rm I}(\mathbf{r}_1,t)\,|^2|\,\mathbf{e}_{\rm II}(\mathbf{r}_2,t)\,|^2\langle q_{\rm I}^\dagger q_{\rm I}\rangle\langle q_{\rm II}^\dagger q_{\rm II}\rangle+|\,\mathbf{e}_{\rm I}(\mathbf{r}_2,t)\,|^2|\,\mathbf{e}_{\rm II}(\mathbf{r}_1,t)\,|^2\langle q_{\rm I}^\dagger q_{\rm I}\rangle\langle q_{\rm II}^\dagger q_{\rm II}\rangle\\&\quad+\{[\mathbf{e}_{\rm I}^*(\mathbf{r}_1,t)\mathbf{e}_{\rm II}(\mathbf{r}_1,t)][\mathbf{e}_{\rm II}^*(\mathbf{r}_2,t)\mathbf{e}_{\rm I}(\mathbf{r}_2,t)]+{\rm c.c.}\}\langle q_{\rm I}^\dagger q_{\rm I}\rangle\langle q_{\rm II}^\dagger q_{\rm II}\rangle\,.\end{aligned} \tag{4.31}$$

While from Eq. (4.30) one concludes that no (standing) interference pattern exists, Eq. (4.31) clearly contains interference terms.

In the two-photon case $m=n=1$, the expectation values $\langle q_{\rm I}^{\dagger 2}q_{\rm I}^2\rangle$ and $\langle q_{\rm II}^{\dagger 2}q_{\rm II}^2\rangle$ vanish. This reflects the simple fact that two photons cannot be detected in a field that contains only one photon. We thus obtain from Eq. (4.31), putting $\omega_{\rm I}=\omega_{\rm II}=\omega$ and $\mathbf{p}_{\rm I}=\mathbf{p}_{\rm II}$ [see G. Richter (1977)],

$$G^{(2)}_{\rm qu}(\mathbf{r}_1,t;\mathbf{r}_2,t)=\frac{4\pi\hbar\omega}{V}\{1+\cos[(\mathbf{k}_{\rm II}-\mathbf{k}_{\rm I})(\mathbf{r}_2-\mathbf{r}_1)]\}\,. \tag{4.32}$$

Obviously, this equation describes an interference effect that is quite similar to that studied in Sec. IV.C [see Eq. (4.20)].

In the general case $m,n\geq 1$, Eq. (4.31) gives us (G. Richter, 1977)

$$G^{(2)}_{\rm qu}(\mathbf{r}_1,t;\mathbf{r}_2,t)=\frac{2\pi\hbar\omega}{V}\{m(m-1)+n(n-1)+2mn+2mn\cos[(\mathbf{k}_{\rm II}-\mathbf{k}_{\rm I})(\mathbf{r}_2-\mathbf{r}_1)]\}\,. \tag{4.33}$$

This result is analogous to Eq. (4.21). From it, we may also draw similar conclusions to those in Sec. IV.D with regard to either large or fluctuating photon numbers.

Now, plane waves certainly represent a very special

form of field distributions, and it appears desirable to extend our considerations to more general situations. To achieve this goal, we can utilize a very elegant tool provided by Titulaer and Glauber (1966), who generalized the usual mode concept. Their procedure is as follows. One starts from a given complete set of conventional modes $\mathbf{g}_k(\mathbf{r},t)$ [e.g., of traveling plane wave type in which case $\mathbf{g}_k(\mathbf{r},t)$ is given by Eq. (4.29)], normalized such that the expansion of $\mathbf{E}^{(+)}(\mathbf{r},t)$ in terms of them reads

$$\mathbf{E}^{(+)}(\mathbf{r},t)=\sum_k \mathbf{g}_k(\mathbf{r},t)q_k \ , \tag{4.34}$$

where q_k is the familiar annihilation operator for a photon in the kth mode. A new set of modes $\mathbf{h}_l(\mathbf{r},t)$ (which are not necessarily monochromatic) is introduced by the definition

$$\mathbf{h}_l(\mathbf{r},t)=\sum_k \gamma_{lk}\mathbf{g}_k(\mathbf{r},t) \ , \tag{4.35}$$

where the coefficients γ_{lk} are the elements of a unitary matrix. Then photon creation and annihilation operators associated with the new modes can be defined through the relations

$$Q_l=\sum_k \gamma_{lk}^* q_k, \quad Q_l^\dagger=\sum_k \gamma_{lk} q_k^\dagger \ . \tag{4.36}$$

Owing to the unitarity of the matrix (γ_{lk}), Q_l and $Q_l^\dagger$ obey, in fact, the familiar commutation relations for photon creation and annihilation operators. Hence the well-known formula for the representation of an n-photon state applies as well to the present case,

$$|\, n\,\rangle_l=\frac{Q_l^{\dagger n}}{(n!)^{1/2}}\,|\,\mathrm{vac}\,\rangle \ , \tag{4.37}$$

where $|\mathrm{vac}\rangle$ is the vacuum state of the radiation field.

Moreover, the expansion of $\mathbf{E}^{(+)}(\mathbf{r},t)$ with respect to the new modes is of precisely the same form as Eq. (4.34),

$$\mathbf{E}^{(+)}(\mathbf{r},t)=\sum_l \mathbf{h}_l(\mathbf{r},t)Q_l \ . \tag{4.38}$$

Hence the usual formalism for the description of quantized fields can be applied with equal success in the case of the generalized modes $\mathbf{h}(\mathbf{r},t)$. In particular, the above results (4.30) and (4.31) remain valid when $\mathbf{e}_\mathrm{I},\mathbf{e}_\mathrm{II}$ are replaced by $\mathbf{h}_\mathrm{I},\mathbf{h}_\mathrm{II}$. The equations (4.32) and (4.33) retain their structure, only the spatial dependence of the interference terms will become more complicated, in general.

It must be stressed, however, that the form of the two wave packets $\mathbf{h}_\mathrm{I}$ and $\mathbf{h}_\mathrm{II}$ cannot be chosen at will, since the above-mentioned unitarity condition must be satisfied. In particular, it is not possible to identify these two modes with two dipole waves originating from different centers.

This difficulty can be overcome, at least partly, by using the following physical argument. Certainly, for the purposes of the observation, only those parts of the interfering waves are relevant that impinge on the receiving plane. Hence we might "cut off" the remaining parts of the fields, e.g., by means of suitable diaphragms (see Fig. 5), without affecting the observation.[3]

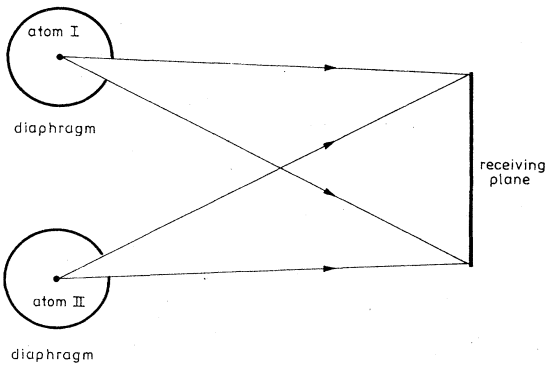

FIG. 5. Geometrical configuration in which the interfering waves do not overlap in $\mathbf{k}$ space.

When those diaphragms absorb, with certainty, any photon falling on them, our device leads to a reduction of the quantum-mechanical wave function in the sense that the outgoing wave, when the source consists of a single atom, either contains the photon or is empty. Hence the waves actually arriving at the receiving plane are in one-photon states. Their spatial structure is determined mainly by the position of the respective diaphragm and the dimensions of its aperture. Now, when the geometry is chosen such that the two interfering waves do not overlap in $\mathbf{k}$ space (see Fig. 5), the unitarity condition is certainly fulfilled, so that the Titulaer-Glauber formalism actually applies.

In more general situations, the calculation becomes more involved, since a reduction to a two-mode problem is no longer possible. Then it is preferable, from the viewpoint of mathematical simplicity, to express the fields through the atomic variables describing their sources, as has been done in Secs. IV.C and IV.D.

F. Historical remarks

The interference effect produced by spontaneously emitting atoms, as it becomes manifest in the modulation of the intensity correlation function, was actually predicted as early as 1954 by Dicke (1954), in the context of cooperative effects in spontaneous emission from many atoms. In fact, Eq. (92) in Dicke's paper, when specialized to the case of two initially excited atoms ($n=2$, $m_0=1$ in Dicke's notation) located at definite positions $\mathbf{r}_\mathrm{I}$ and r_II, respectively, reads

[3]To avoid distortions of the fields by diffraction, the cutoff (solid) angle should be chosen to be a little larger than required by geometrical optics considerations.

$$w(\mathbf{k}_2)=2w_0(\mathbf{k}_2)\left|\frac{\exp(i\Delta\mathbf{k}\mathbf{r}_{\mathrm{I}})+\exp(i\Delta\mathbf{k}\mathbf{r}_{\mathrm{II}})}{2}\right|^2 \quad (\Delta\mathbf{k}=\mathbf{k}_2-\mathbf{k}_1)\ , \tag{4.39}$$

and hence is easily rewritten as

$$w(\mathbf{k}_2)=w_0(\mathbf{k}_2)\{1+\cos[\Delta\mathbf{k}(\mathbf{r}_{\mathrm{II}}-\mathbf{r}_{\mathrm{I}})]\}\ . \tag{4.40}$$

The physical meaning of this result is as follows. It is assumed that the emission of a photon in the direction $\mathbf{k}_1$ has been observed. The quantity $w(\mathbf{k}_2)$ is the probability (per unit time and unit solid angle) that the second photon will be emitted in the direction $\mathbf{k}_2$, and $w_0(\mathbf{k}_2)$ is the corresponding radiation probability for a single isolated excited atom.

Clearly, Dicke's result (4.40) is equivalent to Eq. (4.20). It indicates the striking feature [whose nonclassical nature was stressed by Dicke (1964) in a later paper] that there are special directions for which the emission of the second photon is strictly forbidden. Dicke's interpretation, however, is different from ours. While we consider the appearance of the factor $1+\cos[\Delta\mathbf{k}(\mathbf{r}_{\mathrm{II}}-\mathbf{r}_{\mathrm{I}})]$ as an indication of interference having taken place between two dipole waves emitted independently from one another, Dicke based his argument on the supposition that any photon is emitted in a definite direction by the atom. In this picture, the strong correlation between two emitted photons can be understood only as an effect exerted by the first photon on the atom that will emit the second photon. Obviously, this second atom must, in some way, get information about the position of the first atom and the direction in which it emitted the first photon, in order to be able to "control" the emission processes in accordance with the rule (4.40).

This sounds somewhat mystical (at least in the case of widely separated atoms), as conceded by Dicke himself, who later commented on his result as follows (Dicke, 1964): "One interesting and somewhat paradoxical aspect of the correlation and coherence problem being discussed is the fact that the two radiating atoms could be extremely far apart, many, many wave lengths, and still exhibit this correlation effect. One might naively wonder with such a radiating system, initially in a state for which both atoms are excited, how the one atom would ever know about the existence of the other. It is only because of the presence of the second atom that the radiation distribution pattern for the emission of the second photon depends upon the direction of emission of the first. It should be remembered, however, that both atoms are coupled to the same electromagnetic field. In the process of emitting the first photon, this common coupling results in the excitation of correlation states between the two atoms."

It should be noted that Dicke's supposition is, in fact, not in accordance with the basic interpretational rules for the quantum-mechanical formalism. Quantum mechanics in general forbids us to ascribe any physical property that has become manifest in an observation to the system as, in Einstein's words (Einstein, Podolsky, and Rosen, 1935), an element of reality, before the measurement. Specifically, the direction of emission of a photon by a (localized) excited atom is uncertain as long as no appropriate measurement has been performed. In fact, it is well known that the emitted field has a definite (total) spin (which equals 2 in the case of dipole radiation), and this requires the corresponding quantum-mechanical wave function to be a superposition of linear momentum eigenstates $|\mathbf{k}\rangle$ with virtually all propagation directions coming into play.

The next person to deal with the interference phenomenon under consideration was Fano (1961). Stimulated by the observations of Forrester, Gudmundson, and Johnson (1955) and Brown and Twiss (1956a,1956b), he studied theoretically the intensity correlations produced by two independently emitting atoms. Taking explicitly into account the detection process by representing each detector by an atom initially in its ground state, he calculated, in the framework of quantum-mechanical perturbation theory, the joint probability for both detector atoms to become excited (ionized), ending up with a formula[4] that agrees with Eq. (4.19). He clearly stated that the correlation phenomenon in question is an interference effect. However, he did not mention the specific quantum-mechanical feature of this result that is due to the assumption of precisely two excited atoms being present initially, and hence is, in fact, absent in the above-mentioned observations in which light from thermal sources was the subject of investigation.

Seven years later, Dicke's result (4.40) was rediscovered by Ernst and Stehle (1968), who extended the well-established Wigner-Weisskopf theory (Weisskopf and Wigner, 1930a,1930b) to the case of many atoms. Their physical picture is precisely the same as Dicke's.

To proceed in historical order, I mention a paper by G. Richter (1977), who, amongst other laser physical problems, theoretically investigated the intensity correlations displayed by the superposition of two plane waves with fixed photon numbers, thereby arriving at the results (4.32) and (4.33). Since the fields were statistically independent, Richter's analysis supports the view that the correlations in question have nothing to do with any mutual influence of the emitters via the radiation field, but are rather an optical interference effect.

It should be noted that Dicke (1954,1964) and Ernst and Stehle (1968) were mainly interested in cooperative effects in connection with spontaneous emission, effects which one expects to become noticeable only in situations where the distances between the atoms are less than, or at least comparable with, the wavelength of the emitted light. In spite of this, those authors did not include the dipole-dipole near-zone interaction in their description. This was done later by Steudel (1971), Steudel and

[4]Actually, Fano's result is more general, since it also describes the occurrence of beat notes when the level spacings of the two radiating atoms differ a little.

Richter (1978), and Th. Richter (1979), who, using the master equation derived by Lehmberg (1970) for the atomic variables, took explicit account of both the frequency splitting and the change in the decay rate caused by the interaction of atoms via the radiation field. Extending an investigation by Steudel (1971), Th. Richter (1979) calculated the intensity correlation function $G^{(2)}_{qu}(\mathbf{r}_1,t;\mathbf{r}_2,t+\tau)$ for the two-atom case, which in fact proves to contain near-zone corrections. These are shown to contribute as well to the time-integrated (both over t and τ) intensity correlation function, thereby reducing the strength of the correlation. Interestingly, the single-time correlation function $G^{(2)}_{qu}(\mathbf{r}_1,t;\mathbf{r}_2,t)$ is not affected by the dipole-dipole interaction.

The latest contribution to the theory of intensity correlations produced by independent atomic emitters is due to Mandel (1983). He discussed this problem in the light of optical interference theory, thereby extending his clear analysis to include more general (coherent) initial states of the atoms and fluctuating numbers of atoms. Assuming the atoms to emit independently from each other, he excluded the possibility that there might be a mutual influence during the emission process. Since his results include Dicke's formula (4.40) as a special case, he left no doubt that the intensity correlations, whose nonclassical character he emphasized, are manifestations of photon interference.

V. CONCLUSION

When the term interference is not taken simply as a synonym for the occurrence of an observable interference pattern (as it might be by optical researchers), but is more generally used to denote any effect indicative of the superposition of optical fields, in the sense that their field strengths add rather than their intensities, Dirac's famous statement that "interference between two different photons never occurs" (Dirac, 1930) proves to be false. This has been illustrated in the text by some relevant examples.

First, radiation from coherent sources, such as lasers, has been considered. In view of the close correspondence of such radiation fields with classical electromagnetic waves that possess definite amplitudes and phases, one will not be surprised that in these circumstances interference takes place even in the familiar sense that an interference pattern can be observed. Of more interest is the result of both theoretical and experimental investigations that this interference persists, without loss of visibility, when the laser beams are attenuated so that only a few photons are actually interfering with one another. According to theory, this should be true even when the mean photon number in each of the beams is less than unity. This feature, peculiar as it will appear to common sense, provides an illustrative example of the obstacles to an intuitive understanding of quantum-mechanical predictions.

It is noteworthy that the full equivalence between quantum-mechanical and classical descriptions in the case of coherent fields [more generally, fields that have a P representation, as introduced by Glauber (1963)] rests upon the fact that the number of photons in such fields is uncertain, without any upper bound, in principle. However, when we are dealing with fields produced by a definite number of atoms, the number of photons is certainly bound to be not greater than that of the atoms. (At times long compared to the duration of the emission process, the two numbers obviously will coincide.) Then it can no longer be expected that the quantum-mechanical and the classical predictions will agree.

Actually, a distinct discrepancy is found in the case of two emitting atoms. In these circumstances, no conventional interference pattern can show up, since the atoms radiate with random phases. It is already evident from classical considerations, however, that the corresponding intensity correlation function is not constant in space, but rather exhibits a characteristic modulation, which clearly results from interference. It may come as a surprise that quantum mechanics predicts these correlations to be significantly stronger than those following from the classical theory, stating that for certain positions of the two detectors it is impossible to observe coincidences (each detector being triggered by one photon). Specifically, this happens when the distance between the detectors equals an odd half-number of fringe spacings, with respect to the (fictitious) interference pattern that would occur if the atoms radiated with definite (fixed) phases.

The theoretical analysis reveals that this effect, being nonclassical in its nature, is intimately connected with the simple physical fact that a single atom can emit only one photon and hence is unable to cause two detectors to respond. In other words, when a coincidence is actually counted, one can be sure that the absorbed energy stems from both atoms. This is, however, not so in the classical theory, which is, in fact, at variance with the energy conservation law in the present case, as far as the detection process is concerned. Thus it is ultimately the corpuscular aspect of light that, being properly taken into account in the quantum-mechanical description, enhances the structural order that becomes manifest in the intensity correlations produced by two independently emitted atoms.

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